Bose-Einstein condensation in a dilute, trapped gas at positive temperature

Andreas Deuchert

Institute of Science and Technology Austria (IST Austria)

ICMP @ Montreal
July 2018

Joint work with Robert Seiringer and Jakob Yngvason

Table of contents

Goal: Prove this picture

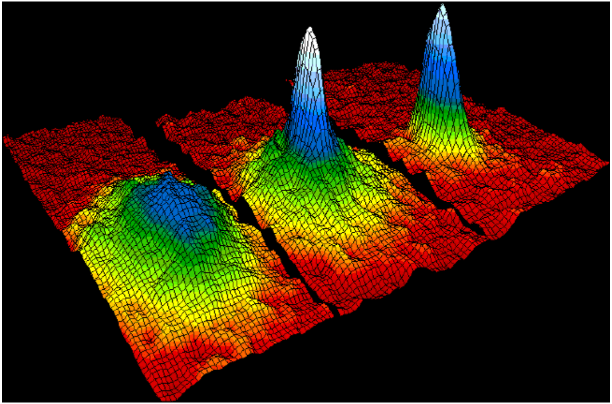


Figure: Emergence of BEC inside the thermal cloud of Rubidium atoms

Anderson et al., Observation of bose-einstein condensation in a dilute atomic vapor, Science 269, 198 (1995)

The ideal Bose gas in the harmonic trap

The expected number of particles in the grand canonical ideal Bose gas governed by the harmonic oscillator Hamiltonian

$$h = -\Delta + \frac{\omega^2 x^2}{4} - \frac{3\omega}{2}$$

is given by

$$\overline{N} = \underbrace{\frac{1}{e^{-\beta\mu_0} - 1}}_{N_0} + \underbrace{\frac{1}{2}\sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{e^{\beta(\omega n - \mu_0)} - 1}}_{N_{\mathrm{th}}}.$$

Here $\beta = T^{-1}$ denotes the inverse temperature and μ_0 is the chemical potential.

The ideal Bose gas in the harmonic trap

The expected number of particles of the grand canonical ideal Bose gas governed by the harmonic oscillator Hamiltonian

$$h = -\Delta + \frac{\omega^2 x^2}{4} - \frac{3\omega}{2}$$

is given by

$$\overline{N} \underbrace{\simeq}_{-\beta\mu_0 \ll \beta\omega \ll 1} \underbrace{\frac{1}{-\beta\mu_0}}_{N_0} + \underbrace{\frac{1}{(\beta\omega)^3} \frac{1}{2} \int_0^\infty \frac{x^2}{e^x - 1} \mathrm{d}x}_{N_{\mathrm{th}}} .$$

Here $\beta = T^{-1}$ denotes the inverse temperature and μ_0 is the chemical potential.

The thermodynamic limit in the trap

The thermodynamic limit in the harmonic trap is defined by

$$N \gg 1 \quad \text{and} \quad (\beta\omega)^{-3} \sim N.$$

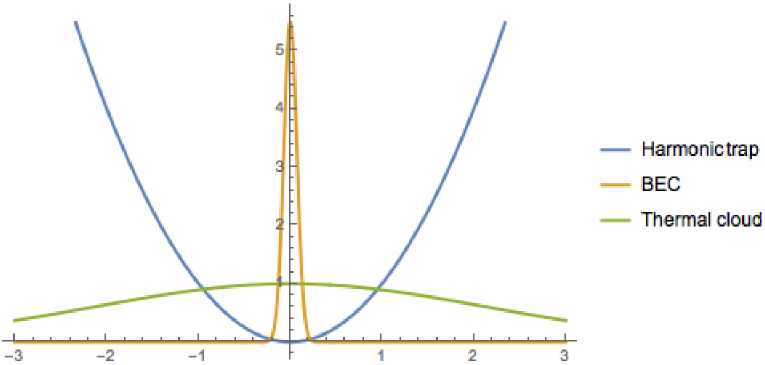
With the critical temperature

$$T_{\mathrm{c}}(N,\omega) = \omega \left(\frac{N}{\zeta(3)} \right)^{1/3},$$

the expected occupation of the ground state in the harmonic trap can be written as

$$\lim \frac{N_0(\beta, N, \omega)}{N} = \lim \left[1 - \left(\frac{T}{T_{\mathrm{c}}} \right)^3 \right]_+ .$$

Length scales



- Length scale condensate: $\omega^{-1/2}$
- Length scale thermal cloud: $\omega^{-1/2}\frac{1}{(\beta\omega)^{1/2}} \gg \omega^{-1/2}$

The model

Hamiltonian with Gross-Pitaevskii scaling:

$$H_N = \sum_{i=1}^{N} \left(-\Delta_i + \frac{\omega^2 x_i^2}{4} - \frac{3\omega}{2} \right) + \sum_{1 \leq i < j \leq N} \omega N^2 v \left(N \omega^{1/2} (x_i - x_j) \right)$$

with $v \geq 0$ such that scattering length is finite. The scattering length of $\omega N^2 v \left(N \omega^{1/2} x \right)$ behaves as

$$a_N \sim \omega^{-1/2} N^{-1} \quad \Rightarrow \quad a_N \ll \omega^{-1/2} \text{ and } N a_N \sim \omega^{-1/2}.$$

The free energy of the gas is given by

$$F(\beta, N, \omega) = -\tfrac{1}{\beta} \ln \left(\mathrm{Tr} \left[e^{-\beta H_N} \right] \right),$$

where the trace is taken only over symmetric functions.

Mathematical literature on dilute Bose gases

- Ground state asymptotics of dilute Bose gas in thermodynamic limit: Dyson '57 (Upper bound hard spheres), Lieb, Yngvason '98 (Lower bound), Lieb, Seiringer, Yngvason '00 (General upper bound)
- Ground state asymptotics of H_N (GP limit): Lieb, Seiringer, Yngvason '00, Lieb, Seiringer '02
- GP limit of rotating Bose gas: Lieb, Seiringer '06, Nam, Rougerie, Seiringer '16
- Bogoliubov theory in GP scaling: Boccato, Brennecke, Cenatiempo, Schlein '18
- Dynamics governend by H_N of BEC : Erdös, Schlein, Yau '09 and '10, Pickl '15, Benedikter, de Oliveira, Schlein '15, Brennecke, Nam, Napiorkowski, Schlein '17
- Free energy asymptotics of dilute Bose gas in thermodynamic limit: Seiringer '08 (Lower bound), Yin '10 (Upper bound)

GP Functional and 1-pdm

Gross-Pitaevskii (GP) functional

$$\mathcal{E}^{\mathrm{GP}}(\phi) = \int_{\mathbb{R}^3} \left(|\nabla\phi(x)|^2 + \left(\tfrac{1}{4}\omega^2 x^2 - \tfrac{3}{2}\omega\right) |\phi(x)|^2 + 4\pi a |\phi(x)|^4 \right) \mathrm{d}x$$

with ground state energy

$$E^{\mathrm{GP}}(N,a) = \inf_{\|\phi\|^2_{L^2(\mathbb{R}^3)}=N} \mathcal{E}^{\mathrm{GP}}(\phi)$$

and minimizer $\phi^{\mathrm{GP}}_{N,a}$. One-particle reduced density matrix (1-pdm) of state Γ_N

$$\gamma_N^{(1)}(x,y) = \mathrm{Tr}[a_x^* a_y \Gamma_N].$$

Theorem: Part 1 (Asymptotics of free energy)

Assumptions:

- v is a nonnegative, radial and measurable function which is integrable outside some finite ball
- Limit: $N \to \infty$, $(\beta\omega)^{-3} \sim N$ and $a_N \sim \omega^{-1/2} N^{-1}$

Notation:

- $F(\beta, N, \omega)$ the canonical free energy related to H_N
- $F_0(\beta, N, \omega) \sim \omega N_{\mathrm{th}}^{4/3}$ the free energy of the ideal gas
- $N_0(\beta, N, \omega)$ expected number of particles in condensate of ideal Bose gas
- $E^{\mathrm{GP}}(N_0, a_N) \sim \omega N_0$ the GP energy

We have

$$\lim \frac{1}{\omega N} \left| F(\beta, N, \omega) - F_0(\beta, N, \omega) - E^{\mathrm{GP}}(N_0, a_N) \right| = 0.$$

Theorem: Part 2 (Asymptotics of 1-pdm)

- State Γ_N with 1-pdm $\gamma_N^{(1)}$ and free energy $\mathcal{F}(\Gamma_N)$
- $\gamma_{N,0}^{(1)}$ denotes 1-pdm of the non-interacting canonical Gibbs state
- φ_0 normalized ground state wavefunction of the harmonic oscillator

For any sequence of states Γ_N with

$$\lim \frac{1}{\omega N} \left| \mathcal{F}_N(\Gamma_N) - F_0(\beta, N, \omega) - E^{\mathrm{GP}}(N_0, a_N) \right| = 0$$

we have

$$\lim \frac{1}{N} \left\| \gamma_N^{(1)} - \left(\gamma_{N,0}^{(1)} - N_0 |\varphi_0\rangle\langle\varphi_0| + |\phi^{\mathrm{GP}}_{N_0,a_N}\rangle\langle\phi^{\mathrm{GP}}_{N_0,a_N}| \right) \right\|_1 = 0.$$

Remarks

- Extension to non-isotropic harmonic trap possible
- Extension to potentials going as $\kappa|x|^\alpha$ for $|x| \to \infty$ with $0 < \alpha < \infty$ and $\kappa > 0$ possible
- All quantities (interacting and non-interacting) can be replaced by grand-canonical versions
- Uniformity in temperature as long as $T \leq CT_{\mathrm{c}}$ for some $C > 0$
- Extension to $d = 2$ with natural replacements possible

Thank you for your attention!