Microscopic Derivation of Ginzburg–Landau Theory and the BCS Critical Temperature Shift in a Weak Homogeneous Magnetic Field

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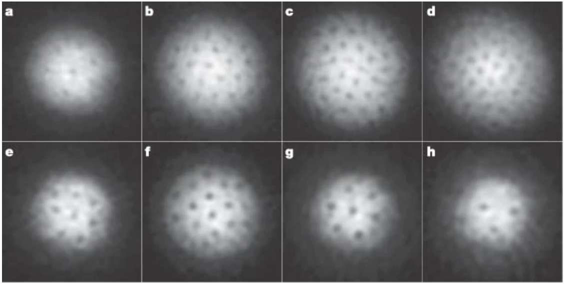
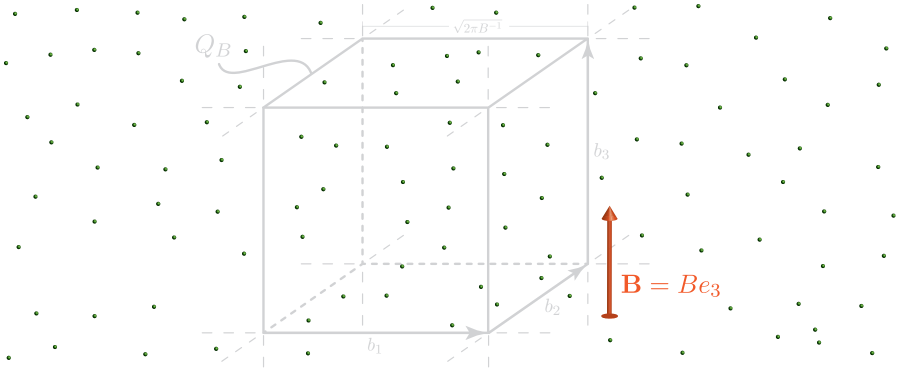
Superfluidity

Vortex lattice of Fermi gas across the **BEC - BCS crossover**.

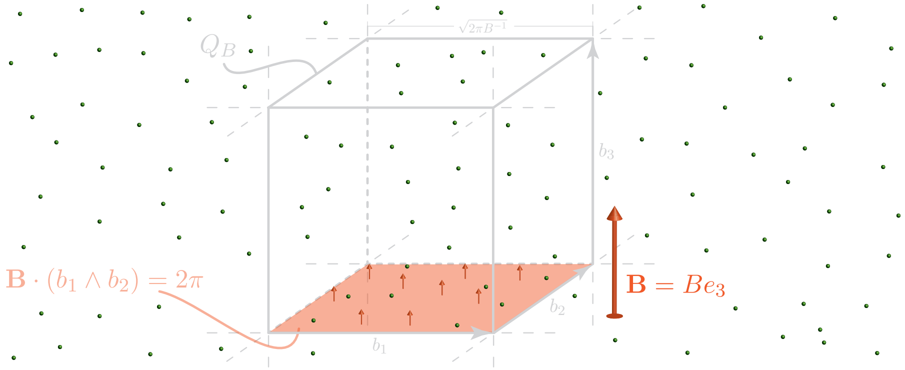


M. W. Zwierlein, J. R. Abo-Shaeer, A. Schirotzek, C. H. Schunck, W. Ketterle, *Vortices and superfluidity in a strongly interacting Fermi gas*, Nature **435**, 1047–1051 (2005)

Setup: Particles in a constant magnetic field



Setup: Particles in a constant magnetic field



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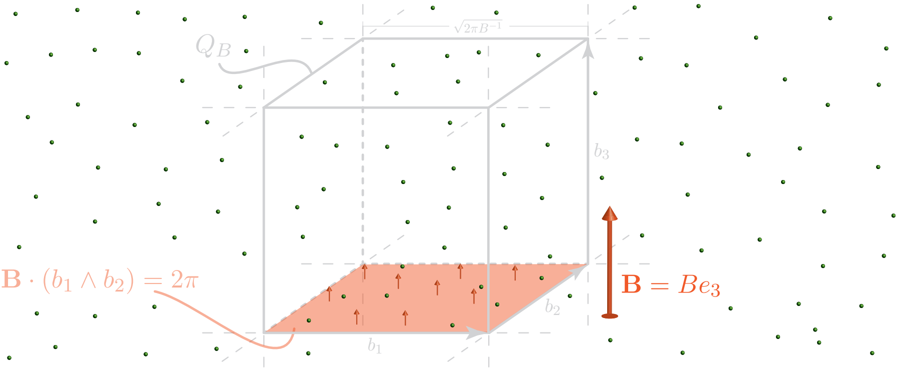


We consider **attractively** interacting fermionic particles in a weak external magnetic field $|\mathbf{B}| \ll 1$.

Fermionic quasi-free states

- States on the fermionic Fock space obeying the Wick theorem are called **quasi-free**.
- Such states are uniquely determined by their **generalized one-particle density matrix** $\Gamma \in \mathcal{L}\left(L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)\right)$, $0 \leq \Gamma \leq 1$, which is of the form

$$\Gamma = \begin{pmatrix} \gamma & \alpha \\ \overline{\alpha} & 1 - \overline{\gamma} \end{pmatrix}.$$

  This, in particular, implies $0 \leq \gamma \leq 1$ and $\alpha\alpha^* \leq \gamma(1-\gamma)$. We assume that $\alpha(x, y) = \alpha(y, x) \Rightarrow$ **s-wave Cooper pairs** (spin-singulett).
- Our states are **gauge-periodic** w.r.t. $\Lambda_B = \sqrt{2\pi B^{-1}}\mathbb{Z}^3$, that is, for $\lambda \in \Lambda_B$ we have

$$\gamma(x + \lambda, y + \lambda) = e^{-\mathrm{i}\frac{\mathbf{B}}{2}\cdot(\lambda\wedge(x-y))}\gamma(x, y),$$
$$\alpha(x + \lambda, y + \lambda) = e^{-\mathrm{i}\frac{\mathbf{B}}{2}\cdot(\lambda\wedge(x+y))}\alpha(x, y).$$

The BCS free energy functional

For gauge-periodic BCS states we define the **BCS free energy functional** by

$$\mathcal{F}^{\mathrm{BCS}}_{\mathbf{B},T}(\Gamma) = \mathrm{Tr}\big[\big((-\mathrm{i}\nabla + \mathbf{A})^2 - \mu\big)\gamma\big] - TS(\Gamma) - \frac{1}{|Q_B|}\int_{Q_B\times\mathbb{R}^3} V(r)|\alpha(X,r)|^2 \mathrm{d}(X,r),$$

where

- $\mathrm{Tr}[A] = \frac{1}{|Q_B|}\mathrm{Tr}_{L^2(Q_B)}[\mathbb{1}(x\in Q_B)A\mathbb{1}(x\in Q_B)]$, that is, we consider **energy per unit volume**, (Q_B is the **unit cell** of the lattice Λ_B)
- $\mathbf{A}(x) = \frac{1}{2}\mathbf{B}\wedge x$ (**Vector potential**),
- $\mu\in\mathbb{R}$, $T\geq 0$ (**chemical potential, temperature**),
- $S(\Gamma) = -\mathrm{Tr}[\Gamma\ln(\Gamma)]$ (**Von Neumann entropy**),
- $r = x - y$, $X = \frac{x+y}{2}$, and $\alpha(X,r)\equiv\alpha(x,y)$, that is, we express α in **relative- and center-of-mass coordinates**.

Normal state, free energy and superconductivity

- The unique minimizer for $V = 0$ is called **normal state** and reads

$$\Gamma_0 = \begin{pmatrix} \gamma_0 & 0 \\ 0 & 1 - \overline{\gamma}_0 \end{pmatrix} \quad \text{with} \quad \gamma_0 = \frac{1}{e^{((-\mathrm{i}\nabla + \mathbf{A}(x))^2 - \mu)/T} + 1}.$$

- The **BCS free energy** is defined by

$$F^{\mathrm{BCS}}(B, T) = \inf_{\Gamma} \{ \mathcal{F}^{\mathrm{BCS}}_{\mathbf{B},T}(\Gamma) - \mathcal{F}^{\mathrm{BCS}}_{\mathbf{B},T}(\Gamma_0) \}.$$

- The system is said to be **superconducting** at temperature T and magnetic field $\mathbf{B}$ if

$$F^{\mathrm{BCS}}(B, T) < 0.$$

Effective models I: Translation-invariant BCS functional

If $\mathbf{B} = 0$ we describe the system by **translation-invariant states**, that is,

$$\gamma(x, y) = \gamma(x - y) \quad \text{and} \quad \alpha(x, y) = \alpha(x - y).$$

In this case it has been shown that there exists a **critical temperature T_c** such that the system is superconducting if $T < T_c$ and in its normal state if $T \geq T_c$.

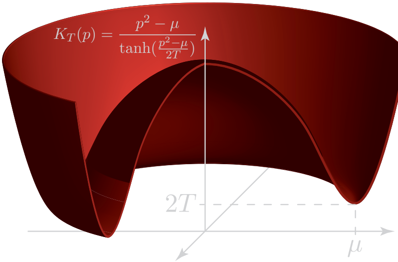
Moreover, T_c can be defined as the **unique value of T** such that the operator

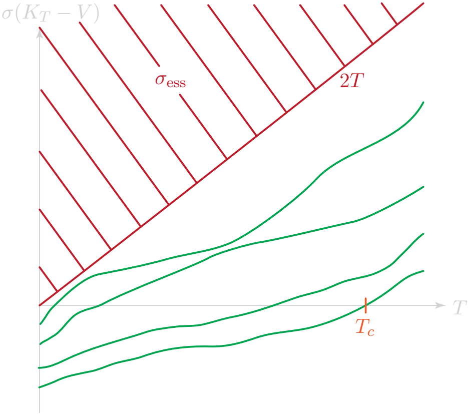
$$K_T - V(x) = \frac{-\Delta - \mu}{\tanh\left(\frac{-\Delta-\mu}{2T}\right)} - V(x)$$

has 0 as its lowest eigenvalue. The **corresponding eigenfunction** will be denoted by α_*.

Critical temperature in translation-invariant case

- $\mathbf{B} = 0$,
- Second variation of BCS functional equals $K_T - V(x)$.





Effective models II: The Ginzburg–Landau functional

We call a function $\Psi : \mathbb{R}^3 \to \mathbb{C}$ **gauge-periodic** if for all $\lambda \in \Lambda_B$ we have

$$\Psi(X+\lambda) = e^{-\mathrm{i}\mathbf{B}\cdot(\lambda\wedge X)}\Psi(X).$$

For such functions and parameters $\Lambda_1, \Lambda_2, \Lambda_3 > 0$, $D \in \mathbb{R}$ the **Ginzburg–Landau functional** is defined by

$$\mathcal{E}^{\mathrm{GL}}_{\mathbf{B},D}(\Psi) = \frac{1}{B^2|Q_B|}\int_{Q_B}\left(\Lambda_1|(-\mathrm{i}\nabla + 2\mathbf{A})\Psi(X)|^2 - DB\Lambda_2|\Psi(X)|^2 + \Lambda_3|\Psi(X)|^4\right)\mathrm{d}X.$$

Its minimal value is the **Ginzburg–Landau energy**

$$E^{\mathrm{GL}}(D) = \inf_{\Psi} \mathcal{E}^{\mathrm{GL}}_{\mathbf{B},D}(\Psi),$$

which, by scaling, does not depend on B.

Selected literature

Translation-invariant BCS functional

- Hainzl, Hamza, Seiringer, Solovej, *The BCS functional for general pair interactions*, CMP **281**, 349 (2008)
- Hainzl, Seiringer, *The Bardeen–Cooper–Schrieffer functional of superconductivity and its mathematical properties*, J. Math. Phys. **57**, 021101 (2016) (review)

BCS functional with external fields, zero magnetic flux through unit cell

- Frank, Hainzl, Seiringer, Solovej, *Microscopic derivation of Ginzburg–Landau theory*, J. Amer. Math. Soc. **25**, 667 (2012)
- Frank, Hainzl, Seiringer, Solovej, *The external field dependence of the BCS critical temperature*, CMP **342**, 189 (2016)

Linearized BCS theory, constant magnetic field

- Frank, Hainzl, Langmann, *The BCS critical temperature in a weak homogeneous magnetic field*, J. Spectr. Theory **9**, 1005 (2019)

Main result part I: Free energy asymptotics

Assumptions:

- V radial function with $(1+|\cdot|^2)V \in L^\infty(\mathbb{R}^3)$,
- V such that $T_c > 0$ in translation-invariant case,
- Zero eigenvalue of $K_{T_c} - V(x)$ is simple.

Theorem (D., Hainzl, Schaub)

There are constants $C > 0$ and $B_0 > 0$ such that for all $0 < B \leq B_0$, we have

$$F^{\mathrm{BCS}}_{\mathbf{B}, T_c(1-DB)} = B^2 \left(E^{\mathrm{GL}}(D) + \mathcal{R} \right)$$

with

$$-CB^{\frac{1}{12}} \leq \mathcal{R} \leq CB.$$

Main result part II: Approximate minimizers

Theorem (continued)

For any approximate minimizer Γ *of* $\mathcal{F}^{\mathrm{BCS}}_{\mathbf{B},T}$ *at* $T = T_{\mathrm{c}}(1 - DB)$ *in the sense that*

$$\mathcal{F}^{\mathrm{BCS}}_{\mathbf{B},T}(\Gamma) - \mathcal{F}^{\mathrm{BCS}}_{\mathbf{B},T}(\Gamma_0) \leq B^2 \left(E^{\mathrm{GL}}(D) + \rho \right)$$

holds for some $\rho \geq 0$*, we have the decomposition*

$$\alpha(X, r) = \alpha_*(r)\Psi(X) + \sigma(X, r)$$

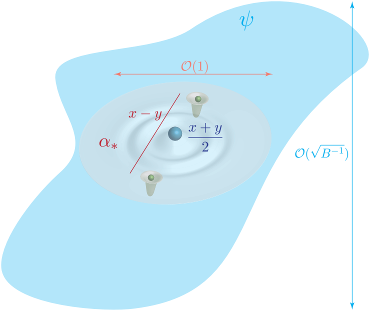
for the **Cooper pair wave function** $\alpha = \Gamma_{12}$*. Here,* σ *satisfies*

$$\frac{1}{|Q_B|} \int_{Q_B \times \mathbb{R}^3} |\sigma(X, r)|^2 \mathrm{d}(X, r) \leq CB^{11/6},$$

α_* **is the zero energy eigenfunction of** $K_{T_{\mathrm{c}}} - V(x)$*, and* Ψ **obeys**

$$\mathcal{E}^{\mathrm{GL}}_{\mathbf{B},D}(\Psi) \leq E^{\mathrm{GL}}(D) + \rho + CB^{1/12}.$$

Separation of scales



- Ψ approximate minimizer of $\mathcal{E}^{\mathrm{GL}}_{\mathbf{B},D}$
- $(K_{T_c} - V(x))\,\alpha_* = 0$

Main result part III: Critical temperature shift

Assumptions: Same as for previous theorem.

Definition: $D_{\mathrm{c}} = \Lambda_1 \inf \sigma_{L^2_{\mathrm{mag}}(Q_1)}((-\mathrm{i}\nabla + e_3 \wedge X)^2)$.

Theorem (D., Hainzl, Schaub)

There are constants $C > 0$ and $B_0 > 0$ such that for all $0 < B \leq B_0$ the following holds:

- *Let $0 < T_0 < T_{\mathrm{c}}$. If the temperature satisfies*

$$T_0 \leq T \leq T_{\mathrm{c}}(1 - B(D_{\mathrm{c}} + CB^{1/2})),$$

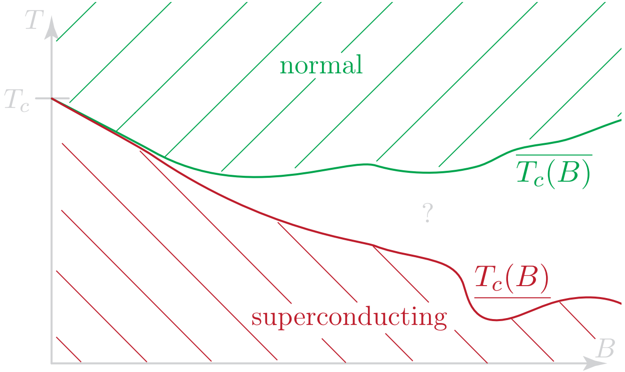
  then we have $F^{\mathrm{BCS}}(B, T) < 0$.

- *If the temperature satisfies*

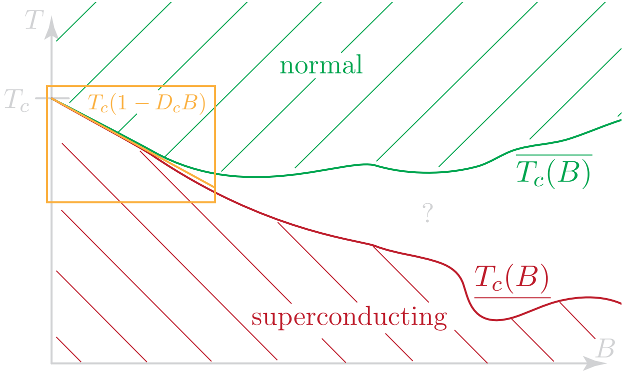
$$T \geq T_{\mathrm{c}}(1 - B(D_{\mathrm{c}} - CB^{1/12})),$$

  then we have $\mathcal{F}^{\mathrm{BCS}}_{\mathbf{B},T}(\Gamma) - \mathcal{F}^{\mathrm{BCS}}_{\mathbf{B},T}(\Gamma_0) > 0$ unless $\Gamma = \Gamma_0$.

Phase Diagram



Phase Diagram



Proof strategy in a nutshell

- Step 1) Learn to carry out very accurate computation with **trial states (Gibbs states)** of the form

$$\Gamma_\Delta = \frac{1}{1+e^{H_\Delta/T}} \text{ with } H_\Delta = \begin{pmatrix} (-\mathrm{i}\nabla + \mathbf{A})^2 - \mu & \Delta \\ \overline{\Delta} & -(-\mathrm{i}\nabla + \mathbf{A})^2 + \mu \end{pmatrix},$$

  and $\Delta(X,r) = 2V(r)\alpha_*(r)\Psi(X)$. $\Rightarrow$ **Upper bound for free energy and lower bound for critical temperature**.

- Step 2) Proof a-priori bound for low energy states close to T_{c}.

- Step 3) For given low energy state Γ, construct Gibbs state Γ_Δ, estimate $\mathcal{F}^{\mathrm{BCS}}_{\mathbf{B},T}(\Gamma)$ from below in terms of $\mathcal{F}^{\mathrm{BCS}}_{\mathbf{B},T}(\Gamma_\Delta)$ (**this is a highly non-trivial thing to do**), and use step 1 to compute $\mathcal{F}^{\mathrm{BCS}}_{\mathbf{B},T}(\Gamma_\Delta)$. **Attention**: Regularity from a-priori bounds not good enough for trial state analysis. **Solution**: Introduce cut-off to obtain more regularity (at the price of additional remainder terms). $\Rightarrow$ **Lower bound for free energy and upper bound for critical temperature**.

Proof techniques I: Upper bound

- **Main novelty of our trial state analysis** is to write

$$-\operatorname{Tr}_0\left[\ln\left(1+e^{-H_\Delta/T}\right)-\ln\left(1+e^{-H_0/T}\right)\right] = \operatorname{Tr}_0\left[\ln\left(\cosh\left(\frac{H_\Delta}{2T}\right)\right)-\ln\left(\cosh\left(\frac{H_0}{2T}\right)\right)\right],$$

  where

$$H_\Delta = \begin{pmatrix} (-\mathrm{i}\nabla+\mathbf{A})^2-\mu & \Delta \\ \overline{\Delta} & -(-\mathrm{i}\nabla+\mathbf{A})^2+\mu \end{pmatrix},$$

  and use a product expansion of $\cosh(x)$.

- Extension of the **phase approximation method**, which has been pioneered in the framework of linearized BCS theory by Frank, Hainzl and Langmann, to our nonlinear setting. The approach makes use of the formula

$$\frac{1}{z-(-\mathrm{i}\nabla+\mathbf{A})^2}(x,y) = e^{\mathrm{i}\frac{\mathbf{B}}{2}\cdot(x\wedge y)} g_B^z(x-y).$$

Proof techniques II: Lower bound

The main novelty of our work are the following a-priori bounds for low energy states:

Theorem (Structure of low energy states)

For all $D_0, D_1 \geq 0$*, there is a constant* $B_0 > 0$ *such that for all* $0 < B_0 \leq B$ *we have the following statement: If* $T - T_{\mathrm{c}} \geq -D_0 B$ *and if* Γ *obeys*

$$\mathcal{F}^{\mathrm{BCS}}_{\mathbf{B},T}(\Gamma) - \mathcal{F}^{\mathrm{BCS}}_{\mathbf{B},T}(\Gamma_0) \leq D_1 B^2,$$

then there exist $\Psi \in H^1_{\mathrm{mag}}(Q_B)$ *and* $\xi \in H^1_{\mathrm{mag}}(Q_B \times \mathbb{R}^3)$ *such that*

$$\alpha(X,r) = \Psi(X)\alpha_*(r) + \xi(X,r),$$

where

$$\sup_{0<B\leq B_0} \|\Psi\|^2_{H^1_{\mathrm{mag}}(Q_B)} \leq C \quad \text{and} \quad \|\xi\|^2_{H^1_{\mathrm{mag}}(Q_B\times\mathbb{R}^3)} \leq CB^2 \left(\|\Psi\|^2_{H^1_{\mathrm{mag}}(Q_B)} + D_1 \right).$$