

GP Upper Band

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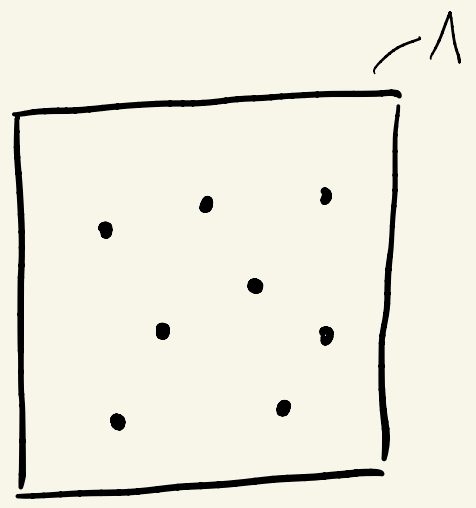
Upper bound for the grand canonical free energy of the Bose gas in the Gross-Pitaevskii limit

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1) Set-up



↳ Torus with side length L .

$$H_N = \int_{\Lambda} a_x^* (-\Delta_x) a_x dx + \frac{1}{2} \int_{\Lambda \times \Lambda} a_x^* a_y^* v_N(x-y) a_y a_x d(x,y)$$

with $v_N(x) = N^2 v(Nx)$ and $v \geq 0$

\downarrow
 $a_N = \frac{a}{N} > 0$ scattering length of v .
radial compactly supported in $L^3(\Lambda)$

Set of probability distributions over rank 1 projections.

States: $S_N = \{ \Gamma \in \mathcal{F}(\mathcal{F}) \mid \Gamma \geq 0, \text{tr} \Gamma = 1, \text{tr} U \Gamma = N \}$

Same N as in definition of Hamiltonian

bosonic Fock space

number of particles

von-Neumann entropy

Gibbs free energy functional: $\mathcal{F}(\Gamma) = \text{tr}[\mathcal{H}_N \Gamma] - \frac{1}{\beta} S(\Gamma)$ with

(grand canonical)

$S(\Gamma) = -\text{tr} \Gamma \ln(\Gamma)$

inverse temperature

Free energy: $F(\beta, N, L) = \inf_{\Gamma \in S_N} \mathcal{F}(\Gamma) \leftarrow$ grand canonical Gibbs state is unique minimizer

$= \phi(\beta, \mu(\beta, N, L), L) + \mu(\beta, N, L) N$

grand canonical potential

→ easier because N also in def. of μ

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Parameters:

·] expected part. numb. N (instead of chem. pot. μ)

·] side length $L = O(1)$ in N

·] inverse temp. $\beta \gtrsim \beta_c = \text{const. } \rho^{-2/3} \sim L^2 N^{-2/3}$ gas and interacting system

\uparrow $\leftarrow \rho = \frac{N}{L^3}$

D., Seiringer '20 (canonical ensemble)

2.) The main result (for the sake of simplicity: $\beta > \beta_c$)

Thm.:
$$F(\beta, N, L) \leq \overset{\substack{\text{density} \\ \downarrow}}{F_0^+}(\beta, N, L) + \delta \bar{\pi} a_N L^3 \left(\overset{\substack{\text{density} \\ \downarrow}}{\rho^2} - \overset{\substack{\text{density of BEC} \\ \downarrow}}{\rho_0^2}} \right) + \overset{\substack{\text{density of BEC} \\ \downarrow}}{F^{\text{BEC}}}(\beta, N, L, a_N)$$

↑
↑
 free energy of thermal cloud free energy of BEC

$$- \frac{1}{2\beta} \sum_{p \neq 0} \left[\frac{16 \bar{\pi} a_N \rho_0}{p^2} - \ln \left(1 + \frac{16 \bar{\pi} a_N \rho_0}{p^2} \right) \right] + o(L^{-2} N^{2/3})$$

↑
 Correction to free energy of thermal cloud from Bogoliubov theory

$$\boxed{\cdot} \quad F_0^+(\beta, N, L) = \frac{1}{\beta} \sum_{p \neq 0} \ln(1 - \exp(-\beta(p^2 - \mu_0))) + \mu_0 N$$

$\mu_0(\beta, N, L) =$ chem. pot. in
ideal gas with
 N particles

↙ We find this by minimizing the
free energy of the condensate

$$\boxed{\cdot} \quad F^{\text{BEC}}(\beta, N_0, L, a_0) = -\frac{1}{\beta} \ln \left(\int_{\mathbb{C}} \exp(-\beta(4\pi a_0 L^{-3} |z|^4 - \mu |z|^2)) dz \right)$$

$$z = x + iy$$

$$dz = \frac{dx dy}{i}$$

exp. # of cond. particles $+ \mu N_0$, where μ is chosen s.t.

$$\int_{\mathbb{C}} |z|^2 \rho(z) dz = N_0 \quad \text{with} \quad \rho(z) = \frac{\exp(-\beta(4\pi a_0 L^{-3} |z|^4 - \mu |z|^2))}{\int_{\mathbb{C}} \exp(\dots) dz}$$

Van der Wouff et al.

← discrete version
for trapped gas

Prop.: Assume that $N_0 \gtrsim N^{5/6+\varepsilon}$, then

$$\overline{F}^{\text{BEC}}(\beta, N_0, L, a_N) = 4\pi a_N L^3 \rho_0^2 + \frac{1}{2\beta} \ln(4\pi a_N \beta L^{-3}) + O(L^{-2} \exp(-cN^\varepsilon)).$$

Cor.: Assume that $N_0 \gtrsim N^{5/6+\varepsilon}$, then

$$\overline{F}(\beta, N, L) \leq \overline{F}_0^+(\beta, N, L) + 4\pi a_N \left(2\rho_0^2 - \rho_0^2 \right) + \frac{1}{2\beta} \ln(4\pi a_N \beta L^{-3}) \quad \text{BEC}$$

$\swarrow \sim L^{-2} N$ $\downarrow L^{-2} N^{2/3} (1 + \ln(N))$

$$\sim L^{-2} N^{5/3} \nearrow \quad - \frac{1}{2\beta} \sum_{p \neq 0} \left[\frac{16\pi a_N \rho_0}{p^2} - \ln \left(1 + \frac{16\pi a_N \rho_0}{p^2} \right) \right] + o(L^{-2} N^{2/3})$$

$\swarrow L^{-2} N^{2/3}$ $\nearrow \text{Bog}$

(*) : Thermodyn. limit : Seiringer '07, Yin '10

GP limit (Proj of cond., can. ensemble) : D., Seiringer '20

Remarks:

← only logarithmic growth with volume

:] The term $\beta \epsilon_C$ vanishes in the thermodynamic limit.

:] The term $\beta \epsilon_0$ converges in the thermodynamic limit to

$$- \frac{16\sqrt{\pi}}{3\beta} (a g_0)^{3/2} \leftarrow \text{compare to: Dapiakowski, Reuvers, Solovej '17}$$

↑ corrections from dispersion relation $\epsilon(p)$ not ϵ_0

:] We have $\int_{\mathbb{R}} |z|^4 \rho(z) dz - \left(\int_{\mathbb{R}} |z|^2 \rho(z) dz \right)^2 \sim N^{5/3}$. — ideal gas: N^2

compare to: Buffet, Pulé '83 (imperfect Bose gas); E.B. Davies '72

3) Heuristics and some ideas of the proof

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\square uncorrelated : $\Gamma_0 = \int_{\mathbb{C}} |z\rangle\langle z| \otimes G(z) g(z) dz$

initial state \downarrow ↪ Gibbs state related to $H^B(z)$

prob. density from BEC effective theory

$\exp(z a_0^* - \bar{z} a_0) |vac\rangle$ with

$$H^B(z) = \sum_{p \neq 0} (\epsilon_p - \mu_0) a_p^* a_p + 4\pi a_0 g_0(\beta, \nu, L) \sum_{p \neq 0} \left[2a_p^* a_p + \frac{z^2}{|z|^2} a_p^* a_{-p}^* + \frac{\bar{z}^2}{|z|^2} a_p a_{-p} \right]$$

↑

temperature-dependent Bogoliubov Hamiltonian, compare with Lee, Yang '58

Spectral Num: $\Gamma_0 = \sum_{\alpha} \lambda_{\alpha} |\psi_{\alpha}\rangle\langle\psi_{\alpha}|$

Final state with correlations: $\Gamma = \sum_{\alpha} \lambda_{\alpha} |\phi_{\alpha}\rangle\langle\phi_{\alpha}|$ with $\phi_{\alpha} = \frac{(1+\beta)\psi_{\alpha}}{\|(1+\beta)\psi_{\alpha}\|}$

and $\mathcal{Z} = \frac{1}{2L^3} \sum_{p, \mu, \nu} \gamma_p a_{u+p}^* a_{v-p}^* a_{\mu} a_{\nu}$, where

$\gamma_p = (\widehat{f_0^{-1}})(p)$ and f_0 solves $\triangle f_0(x) = \frac{V_0(x)}{2} f_0(x)$

↙ full Neumann BC

Compare to: *Yau, Yin '08; Yin '10; Boccato, Brennecke, Cenatiempo, Schlein '13, '20*

Difference: We correlate all interactions in one step.

□ Bogoliubov correction to free energy of thermal cloud

$$\mathcal{E}(p) = \sqrt{p^2 - \mu_0} \sqrt{p^2 - \mu_0 + 16\pi a_0 \rho_0}$$

$$\Phi_{\mathcal{E}}(\beta, \mu_0, L) = \frac{1}{\beta} \sum_{p \neq 0} \ln(1 - \exp(-\beta \mathcal{E}(p)))$$

density-density
interaction
↓

$$= \frac{1}{\beta} \sum_{p \neq 0} \ln(1 - \exp(-\beta(p^2 - \mu_0))) + 8\pi a_0 L^3 (\rho - \rho_0) \rho_0$$

Contributes to
 $\overline{T_0^+}$

$$- \frac{1}{2\beta} \sum_{p \neq 0} \left[\frac{16\pi a_0 \rho_0}{p^2} - \ln \left(1 + \frac{16\pi a_0 \rho_0}{p^2} \right) \right] + o(L^{-2} N^{2/3})$$

↑
new term

Heuristics for condensate effective theory

This is forced upon us
by the result for BEC!

Ansatz: $\Gamma_0 = \int_{\mathbb{C}} |z\rangle\langle z| \rho(z) dz$ with $\int_{\mathbb{C}} |z|^2 \rho(z) dz = N_0$

↑ Correlations implemented before

Hamiltonian: $4\pi a_0 L^{-3} a_0^* a_0^* a_0 a_0$ (condensate has no kinetic energy)

free energy: $4\pi a_0 L^{-3} \int_{\mathbb{C}} |z|^4 \rho(z) dz - \frac{1}{\beta} \underbrace{S(\Gamma_0)}_{\substack{\geq - \int_{\mathbb{C}} \rho(z) \ln(\rho(z)) dz \\ \uparrow \\ \text{Boeziu, Lieb}}}$

$\geq \inf_{\rho \geq 0} \dots = F^{\text{BEC}}(\beta, N_0, L, a_0)$

$\int \rho(z) dz = 1$

$\int |z|^2 \rho(z) = N_0$

c-number substitution for one mode gives $O(L^2)$
Rennecker, Lieb, Seiringer, Yngvason '05

4) Outlook

.] Result with singular potentials coming (Lacro Cap.)

.] Lower bound in mean-field coming (Lacro, Nam)

↳ next step: thermodynamic limit

↳ lower bound in GP