Fourier Series and PDEs (Math 4425)

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4. Separation of Variables

4.1. The wave equation with Dirichlet boundary conditions

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4.1. The wave equation with Dirichlet boundary conditions

We have now all the necessary tools at hand to continue our discussion of the vibrating string in Section 3.1. There we studied the wave equation

$$\begin{cases} \partial_t^2 u(x,t) = c^2 u(x,t) & \text{in } (0,\pi)\times\mathbb{R}_+, \\ u(x,t) = 0 & \text{on } \{0,\pi\}\times\mathbb{R}_+. \end{cases} \qquad (1)$$

We found that the functions

$$u_m(x,t) = \big(A_m \cos(mct) + B_m \sin(mct)\big)\sin(mx) \qquad (2)$$

are solutions to (1). We also showed that if $u_1(x,t)$ and $u_2(x,t)$ are two solutions to (1) then $u(x,t) = u_1(x,t) + u_2(x,t)$ is also a solution (Superposition principle). With this we concluded that a general solution is very likely of the form

$$u(x,t) = \sum_{m=1}^{\infty} \Big(A_m \cos(cmt) + B_m \sin(cmt) \Big) \sin(mx). \quad (3)$$

This looks very much like a Fourier series. Let's try to compute the coefficients A_m and B_m.

We assume that the following initial conditions hold with two periodic[1] functions $f \in C^2([0,\pi], \mathbb{R})$ and $g \in C^1([0,\pi], \mathbb{R})$:

$$u(x,0) = f(x) \quad \text{in } [0,\pi] \times \{t=0\},$$
$$(\partial_t u)(x,0) = g(x) \quad \text{in } [0,\pi] \times \{t=0\}. \quad (4)$$

(1) When we say $f \in C^2([0,\pi], \mathbb{R})$ is periodic we mean that $f(0) = f(\pi)$ and that is twice times continuously differentiable when viewed as a function on the circle (see Section 3).

Insertion of (4) into (3) yields

$$u(x,0) = \sum_{m=1}^{\infty} A_m \sin(mx) = f(x) \tag{5}$$

and

$$\partial_t u(x,0) = \sum_{m=1}^{\infty} c\,m\, B_m \sin(mx) = g(x). \tag{6}$$

Let us first have a close look at (5), which looks almost like a Fourier series. Here almost stands for the fact that f is defined on the interval $[0,\pi]$ and not on an interval of the form $[-L,L]$ with some $L>0$. Since the l.h.s. of (5) is given by a sine series (odd function on $[-\pi,\pi]$) we extend f to $[-\pi,\pi]$ as follows

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in [0,\pi], \\ -f(-x) & \text{if } x \in [-\pi,0). \end{cases} \tag{7}$$

If we now compute the Fourier coefficients of $\tilde{f}$ the resulting Fourier series will equal f on $[0,\pi]$. We know that

$$\tilde{f}(x) = \frac{1}{\sqrt{\pi}} \sum_{m=1}^{\infty} a_m \sin(mx) \quad \text{with}$$

$$a_m = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} \sin(mx)\tilde{f}(x)\,dx$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\pi}^{0} \sin(mx)\big(-f(-x)\big)\,dx$$

$$+ \frac{1}{\sqrt{\pi}} \int_{0}^{\pi} \sin(mx) f(x)\,dx$$

$y=-x$

$$= \frac{-1}{\sqrt{\pi}} \int_{\pi}^{0} \sin(-my)\big(-f(y)\big)\,dy$$

$$+ \frac{1}{\sqrt{\pi}} \int_{0}^{\pi} \sin(mx) f(x)\,dx$$

$$= \frac{2}{\sqrt{\pi}} \int_{0}^{\pi} \sin(mx) f(x)\,dx. \qquad (P)$$

Accordingly, we have $(x \in [0,\pi])$

$$\boxed{\begin{aligned} f(x) &= \sum_{m=1}^{\infty} A_m \sin(mx) \quad \text{with} \\ A_m &= \frac{2}{\pi} \int_0^{\pi} \sin(mx) f(x) . \end{aligned}} \tag{9}$$

Next, we compute the coefficients B_m with the same strategy. We define

$$\tilde{g}(x) = \begin{cases} g(x) & \text{if } x \in [0,\pi] \\ -g(-x) & \text{if } x \in [-\pi,0) \end{cases} \tag{10}$$

on $[-\pi,\pi]$ and expand it into a Fourier series:

$$\tilde{g}(x) = \frac{1}{\sqrt{\pi}} \sum_{m=1}^{\infty} a_m \sin(mx) \quad \text{with}$$

$$a_m = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} \sin(mx)\, \tilde{g}(x)\, dx$$

$$= \frac{2}{\sqrt{\pi}} \int_{0}^{\pi} \sin(mx)\, g(x)\, dx. \tag{11}$$

We compare this expansion for $x \in [0, \pi]$ with (6) and conclude

$$\boxed{\begin{aligned} g(x) &= \sum_{m=1}^{\infty} c_m B_m \sin(mx) \quad \text{with} \\ B_m &= \frac{2}{\pi c_m} \int_0^{\pi} \sin(mx)\, g(x)\, dx \end{aligned}} \tag{12}$$

Finally, we insert our findings for A_m in (9) and B_m in (12) into the formula for u in (3). We have thus shown the following theorem.

Theorem: Let us consider the wave equation

$$\begin{cases} \partial_t^2 u(x,t) = c^2 u(x,t) & \text{in } (0,\pi) \times \mathbb{R}_+, \\ u(x,t) = 0 & \text{on } \{0,\pi\} \times \mathbb{R}_+, \\ u(x,0) = f(x) & \text{on } [0,\pi] \times \{t=0\}, \\ (\partial_t u)(x,0) = g(x) & \text{on } [0,\pi] \times \{t=0\} \end{cases} \tag{13}$$

with two periodic functions $f \in \mathcal{C}^2([0,\pi],\mathbb{R})$ and $g \in \mathcal{C}^1([0,\pi],\mathbb{R})$. The solution to (13) reads

$$u(x,t) = \sum_{m=1}^{\infty} \Big(A_m \cos(cmt) + B_m \sin(cmt) \Big) \sin(mx) \tag{14}$$

with

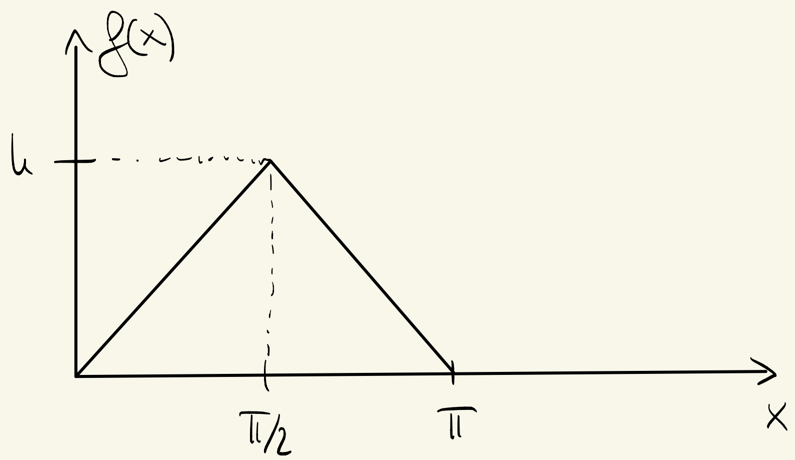
$$A_m = \frac{2}{\pi} \int_0^\pi \sin(mx) f(x) \quad \text{and}$$

$$B_m = \frac{2}{\pi c m} \int_0^\pi \sin(mx) g(x)\, dx. \tag{15}$$

Remark: It can be shown that the expansion in (14) and its first two derivatives w.r.t. t and x converge pointwise.

Example: Let us apply our findings to the example of a plucked string. For the sake of simplicity, we choose $c=1$. As initial conditions we choose $(h>0)$

$$f(x) = \begin{cases} \dfrac{2hx}{\pi} & \text{for } 0 \le x \le \pi/2 \\[2ex] \dfrac{2(\pi-x)h}{\pi} & \text{for } \frac{\pi}{2} \le x \le \pi. \end{cases} \tag{16}$$



and $g(x)=0$ for all $x\in[0,\pi]$. That is, we pluck the string by holding it at height h at $x=p$ and then let go without giving any additional velocity to it. From $g\equiv 0$ we know that $B_n=0$ for all $n\in\mathbb{N}$, and hence

$$u(x,t)=\sum_{n=1}^{\infty} A_n\cos(nt)\sin(nx) \qquad (17)$$

with

$$A_n=\frac{2}{\pi}\int_0^{\pi}\sin(nx)f(x)\,dx \qquad (18)$$

$$=\frac{2}{\pi}\left\{\underbrace{\int_0^{\pi/2}\sin(nx)\,\frac{2hx}{\pi}\,dx}_{①}+\underbrace{\int_{\pi/2}^{\pi}\sin(nx)\,\frac{2(\pi-x)h}{\pi}\,dx}_{②}\right\}.$$

To compute ① we first compute

$$\cdot] \quad \int_0^{\pi/2} e^{ikx}\, x\, dx = \frac{1}{ik} \int_0^{\pi/2} \left(\frac{d}{dx} e^{ikx} \right) x\, dx$$

$$= \frac{1}{ik} \left\{ - \int_0^{\pi/2} e^{ikx}\, dx + \left[e^{ikx}\, x \right]_0^{\pi/2} \right\}$$

$$= \frac{1}{ik} \left\{ - \frac{1}{ik} \underbrace{\int_0^{\pi/2} \left(\frac{d}{dx} e^{ikx} \right) dx}_{= e^{ik\pi/2} - 1} + e^{ik\pi/2} \frac{\pi}{2} \right\}$$

$$= \frac{1}{k^2} \left(e^{ik\pi/2} - 1 \right) + \frac{1}{ik} e^{ik\pi/2} \frac{\pi}{2} \tag{19}$$

and

$$\cdot] \quad \int_{\pi/2}^{\pi} e^{ikx}\, x\, dx = \frac{1}{ik} \left\{ - \int_{\pi/2}^{\pi} e^{ikx}\, dx + \left[e^{ikx}\, x \right]_{\pi/2}^{\pi} \right\}$$

$$= \frac{1}{k^2} \left(e^{ik\pi} - e^{ik\pi/2} \right) + \frac{1}{ik} \left(e^{ik\pi}\pi - e^{ik\pi/2} \frac{\pi}{2} \right). \tag{20}$$

Using (19) and (20) we now compute ① and ②.

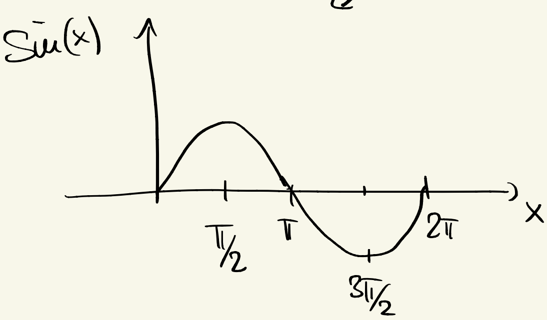
$$① = \int_0^{\pi/2} \sin(nx)\, \frac{2hx}{\pi}\, dx$$

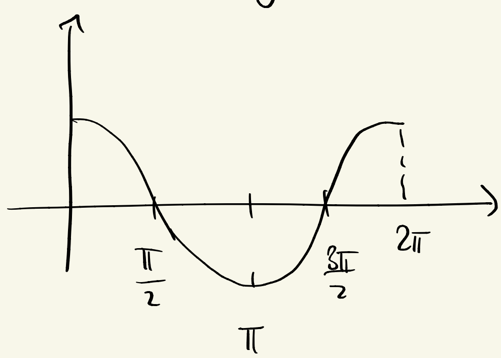
$$= \frac{1}{2i}\, \frac{2h}{\pi} \int_0^{\pi/2} \left(e^{inx} x - e^{-inx} x \right) dx$$

$$= \frac{h}{i\pi} \Big\{ \frac{1}{n^2}\left(e^{in\pi/2} - 1 \right) + \frac{1}{in}\, e^{in\pi/2}\, \frac{\pi}{2}$$

$$- \frac{1}{n^2}\left(e^{-in\pi/2} - 1 \right) + \frac{1}{in}\, e^{-in\pi/2}\, \frac{\pi}{2} \Big\}$$

$$= \frac{h}{i\pi} \Big\{ \frac{1}{n^2} \underbrace{\left(e^{in\pi/2} - e^{-in\pi/2} \right)}_{= 2i \sin(n\pi/2)} + \frac{\pi}{i2n} \underbrace{\left(e^{in\pi/2} + e^{-in\pi/2} \right)}_{2\cos(n\pi/2)} \Big\} \qquad (21)$$





$$\Rightarrow \sin(n\pi/2) = \begin{cases} 0 & \text{if } n \text{ even} \\ (-1)^{\frac{n-1}{2}} & \text{if } n \text{ odd} \end{cases} \qquad \Bigg| \qquad \cos(n\pi/2) = \begin{cases} 0 & \text{if } n \text{ odd} \\ (-1)^{n/2} & \text{if } n \text{ even} \end{cases}$$

We conclude that

$$① = \begin{cases} -\frac{h}{m}(-1)^{m/2} & \text{if } m \text{ even}, \\ \frac{2h}{\pi m^2}(-1)^{\frac{m-1}{2}} & \text{if } m \text{ odd}. \end{cases} \qquad (22)$$

Moreover,

$$② = \int_{\pi/2}^{\pi} \sin(mx)\, \frac{2(\pi - x)h}{\pi}\, dx$$

$$= 2h \underbrace{\int_{\pi/2}^{\pi} \sin(mx)\, dx}_{= \left[-\frac{1}{m}\cos(mx)\right]_{\pi/2}^{\pi} = -\frac{1}{m}\left(\underbrace{\cos(m\pi)}_{(-1)^m} - \cos(m\pi/2)\right)} - \frac{2h}{\pi}\int_{\pi/2}^{\pi} \sin(mx)\, x\, dx$$

$$= \frac{2h}{m}\left(\cos\left(\frac{m\pi}{2}\right) - (-1)^m\right) - \frac{2h}{\pi}\int_{\pi/2}^{\pi} \sin(mx)\, x\, dx \qquad (23)$$

with

$$\frac{2h}{\pi}\int_{\pi/2}^{\pi} \sin(mx)\, x\, dx = \frac{h}{i\pi}\int_{\pi/2}^{\pi}\left(e^{imx}-e^{-imx}\right) x\, dx$$

$$= \frac{h}{i\pi}\left(\frac{1}{m^2}\left(e^{im\pi}-e^{im\pi/2}\right)+\frac{1}{im}\left(e^{im\pi}\pi - e^{im\pi/2}\frac{\pi}{2}\right)\right)$$

$$-\frac{h}{i\pi}\left(\frac{1}{m^2}\left(e^{-im\pi}-e^{-im\pi/2}\right)-\frac{1}{im}\left(e^{-im\pi}\pi - e^{-im\pi/2}\frac{\pi}{2}\right)\right)$$

$$= \frac{h}{i\pi m^2}\left[\underbrace{e^{im\pi}-e^{-im\pi}}_{2i\sin(m\pi)=0} - \underbrace{e^{im\pi/2}+e^{-im\pi/2}}_{-2i\sin(m\pi/2)}\right]$$

$$-\frac{h}{m\pi}\left(2\pi\underbrace{\cos(m\pi)}_{=(-1)^m} - \pi\cos(m\pi/2)\right)$$

$$= -\frac{2h}{\pi m^2}\sin(m\pi/2) - \frac{h}{m\pi}\left(2\pi(-1)^m - \pi\cos\left(\frac{m\pi}{2}\right)\right) \qquad (24)$$

Insertion of (24) into (23) yields

$$② = \frac{2h}{n}\left(\cos\left(\frac{n\pi}{2}\right) - (-1)^n\right) + \frac{2h}{\pi n^2}\sin(n\pi/2)$$

$$+ \frac{h}{n\pi}\left(2\pi(-1)^n - \pi\cos\left(\frac{n\pi}{2}\right)\right)$$

$$= \frac{h}{n}\cos(n\pi/2) + \frac{2h}{\pi n^2}\sin\left(\frac{n\pi}{2}\right)$$

$$= \begin{cases} \frac{h}{n}(-1)^{n/2} & \text{if } n \text{ even} \\ \frac{2h}{\pi n^2}(-1)^{\frac{n-1}{2}} & \text{if } n \text{ odd}. \end{cases} \tag{25}$$

In the last step we insert (22) and (25) into (18), which gives

$$A_n = \frac{2}{\pi}\left(① + ②\right) = \begin{cases} 0 & \text{if } n \text{ even}, \\ \frac{8h}{\pi^2 n^2}(-1)^{\frac{n-1}{2}} & \text{if } n \text{ odd}. \end{cases} \tag{26}$$

The solution the wave equation therefore reads

$$\boxed{u(x,t) = \frac{8h}{\pi^2}\sum_{m=1}^{\infty}(-1)^m \frac{1}{(2m+1)^2}\cos\left((2m+1)t\right)\sin\left((2m+1)x\right)} \tag{27}$$

Remark: As a final remark, we should note an unsatisfactory aspect of the solution to this problem, which, however, is in the nature of things. Since the initial condition f is not twice differentiable, neither is the solution u given by (27). Hence u is not truly a solution to the wave equation: while $u(x,t)$ does represent the position of the plucked string, it does not satisfy the PDE we set out to solve! This state of affairs may be understand properly only if we realize that u does solve the equation, but in an appropriate generalized sense. A better understanding of this phenomenon requires ideas relevant to the study of "weak solutions" and the theory of "distributions". Both topics are beyond the scope of this lecture.

4.2 The heat equation with Neumann boundary conditions

The goal of this section is to apply the idea of separation of variables that was so successful for the wave equation to the heat equation. At the same time we also will learn how to with with other boundary conditions. Because of this, we study the heat equation with Neumann boundary conditions. It is needless to say that the same approach also works for Dirichlet BCs.

The heat equation we will study reads

$$\begin{cases} \partial_t u(x,t) = k\, \partial_x^2 u(x,t) & \text{in } (0,\pi) \times \mathbb{R}_+ \\ (\partial_x u)(x,t) = 0 & \text{on } \{0,\pi\} \times \mathbb{R}_+ \end{cases} \qquad (28)$$

with some $k > 0$.

As in case of the wave equation we want to find all solutions. We start by looking for special solutions of the form

$$u(x,t) = \varphi(x)\psi(t). \tag{29}$$

Insertion into the first equation in (28) yields

$$\varphi(x)\dot{\psi}(t) = k\,\varphi''(x)\psi(t)$$

$$\Leftrightarrow \qquad \frac{\dot{\psi}(t)}{k\,\psi(t)} = \frac{\varphi''(x)}{\varphi(x)}. \tag{30}$$

(30) can be satisfied only if both sides equal the same constant $\lambda \in \mathbb{R}$. In the exercises you showed that all solution (λ, φ) to the problem

$$\begin{cases} \varphi''(x) = \lambda\,\varphi(x) \\ \varphi'(0) = 0 = \varphi'(\pi) \end{cases} \tag{31}$$

are of the form

$$\varphi_m(x) = A\cos(mx), \quad A \in \mathbb{R}$$

$$\lambda_m = -m^2 \quad \text{with} \quad m \in \mathbb{N}_0. \tag{32}$$

For these values of λ we now solve

$$\dot{\psi}(t) = k\lambda_m \psi(t)$$

$$\Rightarrow \quad \psi(t) = \psi(0)\, e^{k\lambda_m t}. \tag{33}$$

We conclude that the functions

$$\boxed{u_m(x,t) = A_m \cos(mx)\, e^{-km^2 t}, \quad m \in \mathbb{N}_0} \tag{34}$$

are all solutions to (28). Since also the heat equation satisfies the superposition principle we conclude that a general solution to (28) should be of the form

$$\boxed{u(x,t) = \sum_{m=0}^{\infty} A_m \cos(mx)\, e^{-k m^2 t}} \tag{35}$$

Let us now add the initial condition $u(x,0) = f(x)$ with some $f : [0,\pi] \to \mathbb{R}$ in order to obtain a unique solution. As in the case of the wave equation we need to find a way to compute the coefficients $\{A_m\}_{m=0}^{\infty}$ for given f.

We need to choose A_m s.t.

$$u(x,0) = \sum_{m=0} A_m \cos(mx) \stackrel{!}{=} f(x). \tag{36}$$

Since $\cos(mx)$ is an even function on $[-\pi,\pi]$ we define the even extension $\tilde{f}$ of f by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in [0,\pi] \\ f(-x) & \text{if } x \in [-\pi,0), \end{cases} \tag{37}$$

compute the Fourier expansion of $\tilde{f}$ and restrict it to $[0,\pi]$ to obtain a series representation of f:

$$\tilde{f}(x) = \frac{a_0}{\sqrt{2\pi}} + \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} a_n \cos(nx) \qquad (38)$$

with

$$a_0 = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \tilde{f}(x)\,dx = \sqrt{\frac{2}{\pi}} \int_0^{\pi} f(x)\,dx$$

and

$$a_n = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} \cos(nx)\tilde{f}(x)\,dx$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\pi} \cos(nx) f(x)\,dx. \qquad (39)$$

We conclude that

$$\boxed{f(x) = \sum_{n=0}^{\infty} A_n \cos(nx) \quad \text{with} \quad A_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx, \quad A_n = \frac{2}{\pi}\int_0^{\pi} \cos(nx) f(x)\,dx.} \qquad (40)$$

This allows us the conclude that the solution to the heat equation in (28) subject to the initial condition $u(x,0) = f(x)$ is given by

$$\boxed{\begin{gathered} u(x,t) = \sum_{n=0}^{\infty} A_n \cos(nx)\, e^{-\kappa n^2 t} \\ \text{with} \\ A_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx, \quad A_n = \frac{2}{\pi}\int_0^{\pi} \cos(nx)\, f(x)\,dx. \end{gathered}} \qquad (41)$$

That is, we obtained a general solution to the heat equation in the same way as we did for the wave equation. These two studies taught us how to treat Dirichlet ($u(x,t) = 0$ for $x \in \{0,\pi\}$) and Neumann ($(\partial_x u)(x,t) = 0$ for $x \in \{0,\pi\}$) boundary conditions.

On the next exercise sheet you will compute a concrete example. You will also solve the heat equation with another boundary condition.

4.3. The heat equation with Robin boundary conditions

In this section we consider the heat equation ($k>0$)

$$\begin{cases} \partial_t u(x,t) = k\,\partial_x^2 u(x,t) & \text{on } (0,2)\times\mathbb{R}_+ \\ u(0,t) = 0, & \\ (\partial_x u)(2,0) = -u(2,0) & \end{cases} \tag{42}$$

with mixed Dirichlet and Robin boundary conditions. As we will see things are a little more complicated in this case. Please note that the heat equation in (42) still satisfies the superposition principle.

It therefore makes sense to again look for special solutions of the form $u(x,t) = \varphi(x)\,\psi(t)$. In this case the spatial function $\varphi(x)$ needs to solve the eigenvalue equation (compare to p.17)

$$\begin{cases} \varphi''(x) = \lambda \varphi(x), \\ \varphi(0) = 0, \\ \varphi'(2) = -\varphi(2). \end{cases} \tag{43}$$

A general solution to (43) is of the form

$$\varphi(x) = c_1 e^{k_1 x} + c_2 e^{k_2 x},$$

where k_1, k_2 are the two solution to the characteristic equation

$$k^2 = \lambda. \tag{44}$$

As on the exercise sheets one easily checks that there are no solutions if $\lambda > 0$. If $\lambda = 0$ all solutions are of the form $ax+b$ and also they do not satisfy the boundary conditions.* It remains to consider the case $0 > \lambda = -m^2$ with

(*) (except if $a, b, \lambda = 0$)

$m>0$. Using $\varphi(0)=0$ we conclude that

$$\varphi(x) = A\sin(kx) \tag{45}$$

with two constants $A, k \in \mathbb{R}\setminus\{0\}$.

Next, we use the second boundary condition to solve for k:

$$\begin{aligned} \varphi'(2) &= -\varphi(2) \\ \Leftrightarrow \quad Ak\cos(2k) &= -A\sin(2k) \\ \Leftrightarrow \quad k &= -\tan(2k). \end{aligned} \tag{46}$$

Plotting the functions in (46) we can check that the solutions are <u>not</u> equally spaced as seen in earlier problems. Moreover, we cannot compute them exactly by hand. Here is a table of the first ten solutions to (46).

n	k_n	n	k_n
1	1.144465	6	8.696622
2	2.54345	7	10.258761
3	4.048082	8	11.823162
4	5.586353	9	13.389044
5	7.138177	10	14.955847

If we want to solve (42) we need to compute these values numerically. Let's assume we have done that, how can we compute the solution to (40)?

We can still write the general solution to (40) in a series expansion of the form (solve eq. for T to find the time dependence!)

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} A_n \, e^{-K k_n t} \sin(k_n x).} \qquad (47)$$

Let us again impose the initial condition $u(x,0) = f(x)$

and try to compute the coefficients A_n. We have

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(k_n x) \overset{!}{=} f(x). \tag{48}$$

Since the k_n are not evenly spaced we cannot simply apply a Fourier expansion to compute them.

But we can do the following. We treat the functions as basis functions w.r.t. the inner product (defined on real functions)

$$\langle f, g \rangle_2 = \int_0^2 f(x) g(x) \, dx. \tag{49}$$

First, we need to check that orthogonality holds, that is, we need to show that for $n \neq m$ we have

$$\int_0^2 \sin(k_n x) \sin(k_m x) \, dx \overset{!}{=} 0. \tag{50}$$

We have already learned how to compute integrals of this form (note that k_n and k_m are <u>not</u> integers), and I therefore only state the result:

$$\int_0^2 \sin(k_n x)\sin(k_m x)\,dx$$

$$= \frac{k_m \sin(2k_n)\cos(2k_m) - k_n \sin(2k_m)\cos(2k_n)}{k_n^2 - k_m^2}$$

$$= \frac{k_m k_n \cos(2k_m)\cos(2k_n)}{k_n^2 - k_m^2}\left(\underbrace{\frac{\sin(2k_n)}{k_n\cos(2k_n)}}_{=\frac{\tan(k_n)}{k_n} = -1} - \underbrace{\frac{\sin(k_m)}{k_m\cos(k_m)}}_{\frac{\tan(k_m)}{k_m} = -1}\right)$$

$k = -\tan(2k) \leftarrow (46)$

$$= 0. \tag{51}$$

That is, the functions $\sin(k_n x)$ and $\sin(k_m x)$ with $n \neq m$ are orthogonal w.r.t. the inner product $\langle \cdot, \cdot \rangle_2$!

Let us define the numbers

$$\alpha_m = \int_0^2 \sin^2(k_m x)\, dx. \tag{52}$$

We conclude that the function

$$\varphi_m(x) = \frac{1}{\sqrt{\alpha_m}} \sin(k_m x)$$

are an orthonormal w.r.t. $\langle \cdot, \cdot \rangle_2$, that is,

$$\langle \varphi_m, \varphi_n \rangle = \delta_{n,m}. \tag{53}$$

We also have:

Theorem: Let $\{(\lambda_n, \psi_n)\}_{n=1}^{\infty}$ be the normalized (i.e. $\langle \psi_m, \psi_m \rangle = 1$) solutions to the equation

$$\Psi''(x) = \lambda \Psi(x) \quad , \quad x \in [a,b] \tag{54}$$

with either Dirichlet ($\Psi(y) = 0$, $y \in \{a,b\}$), Neumann ($\Psi'(y) = 0$, $y \in \{a,b\}$), or Robin boundary conditions ($\Psi'(y) = c\Psi(y)$, $y \in \{a,b\}$, $c \in \mathbb{R}$) or mixtures thereof (e.g. Dirichlet at $y = a$ and Neumann at $y = b$). Then the functions are a Schauder basis for Riemann integrable (real- or complex-valued) functions on $[a,b]$.

Let's go back to the problem of finding the coefficients for the expansion (Eq. (48))

$$\sum_{m=1}^{\infty} A_m \underbrace{\sin(k_m x)}_{=\sqrt{\alpha_m}\,\varphi_m(x)} = f(x). \tag{55}$$

If (55) holds then we have

$$\sum_{m=1}^{\infty} A_m \sqrt{\alpha_m}\, \varphi_m = f$$

$$\Rightarrow \quad \left\langle \varphi_k, \sum_{m=1}^{\infty} A_m \sqrt{\alpha_m}\, \varphi_m \right\rangle = \langle \varphi_k, f \rangle$$

$$\parallel$$

$$\sum_{m=1}^{\infty} A_m \sqrt{\alpha_m} \underbrace{\langle \varphi_k, \varphi_m \rangle}_{\delta_{k,m}} = A_k \sqrt{\alpha_k}. \qquad (56)$$

We conclude that

$$\boxed{\begin{aligned} A_n &= \frac{1}{\sqrt{\alpha_n}} \langle \varphi_n, f \rangle \\ &= \frac{1}{\alpha_k} \int_0^2 \sin(k_n x)\, f(x)\, dx. \end{aligned}} \qquad (57)$$

If we insert this choice for A_n with the values for k_n that have previously been found into (47) we obtain a solution to the heat equation in (42) that satisfies the initial condition $u(x,0) = f(x)$.

4.4. The heat equation with a source term

In this section we consider the heat equation

$$\begin{cases} \partial_t u(x,t) = \partial_x^2 u(x,t) + Q(x) & \text{in } (0,\pi)\times\mathbb{R}_+ \\ u(x,t) = 0 & \text{on } \{0,\pi\}\times\mathbb{R}_+ \\ u(x,0) = f(x) & \text{on } [0,\pi]\times\{t=0\} \end{cases} \qquad (SP)$$

with the source term $Q(x)$. Note that we have chosen Dirichlet boundary conditions.

Let us recall how we solved inhomogeneous problems earlier, e.g., in case of the transport equation. The first step was to solve the homogeneous problem, which gave us what we called a propagator. Then we used the propagator to write down the solution to the inhomogeneous problem (please read this part for the transport equation again before you continue reading this).

We already solved the homogeneous problem with Neumann boundary conditions. To change the boundary conditions we need to replace the eigenvalues and eigenfunctions of the eigenvalue problem

$$\begin{cases} \varphi''(x) = \lambda \varphi(x) \\ \varphi'(0) = 0 = \varphi'(\pi) \end{cases} \tag{59}$$

(see page 17) by those of the same problem with Dirichlet boundary conditions

$$\varphi(0) = 0 = \varphi(\pi). \tag{60}$$

These solutions have been computed in Exercise 15. They read

$$\lambda_n = -n^2 \quad \text{with} \quad n \in \mathbb{N},$$

$$\varphi_n(x) = A \sin(nx) \quad \text{with} \quad A \in \mathbb{R}. \tag{61}$$

The solution to (58) with $Q = 0$ is therefore given by

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(nx)\, e^{-n^2 t} \tag{62}$$

with

$$A_n = \frac{2}{\pi} \int_0^{\pi} \sin(nx) f(x)\, dx, \tag{63}$$

compare with the analysis in Section 4.2.

Let us give another interpretation of this. Let $f \in \mathbb{R}$ be an integrable function. We already introduced the map $\mathcal{F}: R \to \ell^2(\mathbb{Z}, \mathbb{C})$ that maps f to its Fourier coefficients. Here, we define $\mathcal{F}$ s.t. it maps the function f to the coefficients $\{A_n\}_{n=1}^{\infty}$, that is,

$$\mathcal{F}(f)(n) = A_n \quad \text{with}$$

$$A_n = \frac{2}{\pi} \int_0^{\pi} \sin(nx) f(x)\, dx. \tag{64}$$

We also define the inverse map $\mathcal{F}^{-1}$ by

$$\mathcal{F}^{-1}\left(\{A_n\}_{n=1}^{\infty}\right)(x) = \sum_{n=1}^{\infty} A_n \sin(nx). \tag{65}$$

Motivated by (62), we also define the map

M for multiplication operator

$$\mathcal{M}_t : \ell^2(\mathbb{N}, \mathbb{C}) \to \ell^2(\mathbb{N}, \mathbb{C}),$$

$$\mathcal{M}_t(A_n) = A_n e^{-n^2 t}. \tag{66}$$

With these definitions at hand, we can write

(62) as follows

$$u(x,t) = \mathcal{F}^{-1}\Big(M_t\big(\mathcal{F}(f)(n)\big)\Big)(x),$$

$$\Leftrightarrow \quad \underbrace{u(t)}_{} = \underbrace{\mathcal{F}^{-1} M_t \mathcal{F}}_{} \underbrace{f}_{}. \qquad (67)$$

here we view $u(t)$ as a vector and therefore omit the x-dependence

This is the propagator. It is a linear map that maps the initial condition to the solution to (SP) at time t.

initial condition

In the following we denote the propagator by P_t, that is,

$$(P_t f)(x) = \sum_{n=1}^{\infty} A_n \sin(nx)\, e^{-n^2 t}. \qquad (68)$$

This ends step 1, we have written (62) in terms of the propagator P_t.

Next, we recall how we solved the ODE

$$\begin{cases} \dot{x}(t) = a\,x(t) + b, \\ x(0) = \bar{x}. \end{cases} \tag{69}$$

The solution to the homogeneous problem $(b=0)$ is

$$x(t) = e^{at}\,\bar{x} \tag{70}$$

and that to the inhomogeneous reads

$$x(t) = e^{at}\,\bar{x} + \int_0^t e^{a(t-s)}\, b\, ds. \tag{71}$$

If we define the propagator P_t by

$$P_t\,\bar{x} = e^{at}\,\bar{x}, \tag{72}$$

(71) can be written as

$$x(t) = P_t\,\bar{x} + \int_0^t P_{(t-s)}\, b\, ds. \tag{73}$$

Let's guess a solution to (58) that is inspired by (73). We define the function (vector notation!)

$$u(t) = P_t f + \int_0^t P_{(t-s)} Q \, ds$$

$$\Leftrightarrow \quad u(x,t) = \sum_{n=1}^{\infty} A_n \sin(nx) e^{-n^2 t} + \int_0^t \sum_{n=1}^{\infty} B_n \sin(nx) e^{-n^2(t-s)} \, ds$$

with $A_n = \frac{2}{\pi} \int_0^\pi \sin(nx) f(x) \, dx$ and

$$B_n = \frac{2}{\pi} \int_0^\pi \sin(nx) Q(x) \, dx. \tag{74}$$

Let's check that $u(x,t)$ is indeed a solution to (58). We first check that it satisfies the first equation in (58):

$$\bullet \quad \partial_t u(x,t) = \sum_{n=1}^{\infty} -n^2 A_n \sin(nx)\, e^{-n^2 t}$$

$$+ \underbrace{\sum_{n=1}^{\infty} B_n \sin(nx)}_{= Q(x)} + \int_0^t \sum_{n=1}^{\infty} B_n \sin(nx) (-n^2)\, e^{-n^2(t-s)}\, ds \tag{75}$$

$$\bullet \quad \partial_x^2 u(x,t) = \sum_{n=1}^{\infty} A_n \underbrace{\left[\partial_x^2 \sin(nx)\right]}_{= -n^2 \sin(nx)} e^{-n^2 t}$$

$$+ \int_0^t \sum_{n=1}^{\infty} B_n \underbrace{\left[\partial_x^2 \sin(nx)\right]}_{= -n^2 \sin(nx)} e^{-n^2(t-s)}\, ds \tag{76}$$

We conclude that

$$\partial_t u(x,t) = \partial_x^2 u(x,t) + Q(x) \tag{77}$$

holds.

Next, we check the boundary condition:

$$\cdot]\ u(0,t) = \sum_{n=1}^{\infty} A_n \underbrace{\sin(0)}_{=0} e^{-n^2 t}$$

$$+ \int_0^t \sum_{n=1}^{\infty} B_n \underbrace{\sin(0)}_{=0} e^{-n^2(t-s)}\, ds = 0 \qquad (78)$$

The same computation shows $u(\pi,t) = 0$.

Finally, we check that the initial condition is satisfied:

$$\cdot]\quad u(x,0) = \sum_{n=1}^{\infty} A_n \sin(nx) \underset{\uparrow}{=} f(x). \qquad (41)$$

see definition of A_n

That is, the function in (74) is indeed a solution to (58).

4.5. Laplace's equation

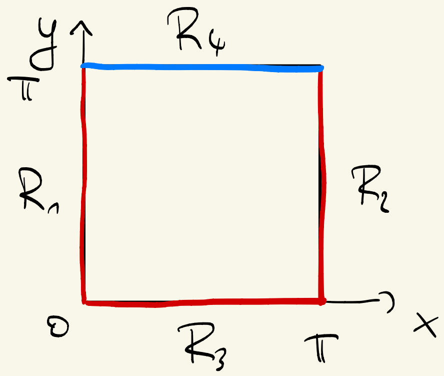
In this section we show how the technique of separation of variables can be used to solve Laplace's equation. Since we do not have much time left we discuss one concrete case.

We want to solve the equation

$$\partial_x^2 u(x,y) + \partial_y^2 u(x,y) = 0 \quad \text{on } (0,\pi)\times(0,\pi) \qquad (42)$$

with the boundary condition

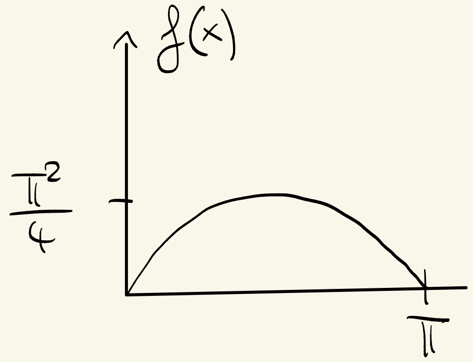
$$u(x,0) = 0, \quad u(x,\pi) = f(x)$$
$$u(0,y) = 0, \quad u(\pi,y) = 0 \qquad (43)$$



Here u equals f

Here u equals 0

If you prefer something even more concrete you can assume that $f(x) = x(\pi - x)$.



As before we will first try to find special solutions of the form $\varphi(x)\psi(y)$ and then expand a general solution in terms of the special solutions. Finally, we will try to incorporate the boundary conditions.

<u>Step 1:</u> Let's assume $u(x,y) = \varphi(x)\psi(y)$ and insert this ansatz into (42). We find

$$\frac{\varphi''(x)}{\varphi(x)} = -\frac{\psi''(y)}{\psi(y)} \tag{44}$$

and we conclude that

$$\frac{\varphi''(x)}{\varphi(x)} = \lambda = -\frac{\psi''(y)}{\psi(y)} \qquad (45)$$

holds with some constant $\lambda \in \mathbb{R}$. For our ansatz the boundary conditions read

$$\varphi(x)\psi(0) = 0 \quad , \quad \varphi(x)\psi(\pi) = f(x) ,$$

$$\varphi(0)\psi(y) = 0 \quad , \quad \varphi(\pi)\psi(y) = 0 . \qquad (46)$$

We conclude that $\psi(0) = 0$, $\varphi(0) = 0$, $\varphi(\pi) = 0$.

The remaining boundary condition will be used later.

Let us solve the equation for φ:

$$\begin{cases} \varphi''(x) = \lambda \varphi(x) \\ \varphi(0) = 0 = \varphi(\pi) \end{cases} \qquad (47)$$

From Exercise 15 we know that the solutions read (λ_n, φ_n) with

$$\lambda_n = -n^2, \quad \varphi_n(x) = A_n \sin(nx), \quad n \in \mathbb{N}. \tag{48}$$

What about the equation for ψ? We use (45) and the boundary condition $\psi(0) = 0$:

$$\begin{cases} \psi''(y) = n^2 \psi(y), \\ \psi(0) = 0. \end{cases} \tag{49}$$

As seen already several times, a solution to the first equation in (49) is of the form

$$\psi_n(y) = c_1 e^{nx} + c_2 e^{-nx}. \tag{50}$$

We want

$$\psi_n(0) = c_1 + c_2 = 0 \iff c_2 = -c_1, \tag{51}$$

and hence

$$\varphi_n(y) = \frac{c_n}{2}\left(e^{nx} - e^{-nx}\right)$$

$$= c_n \sinh(nx). \tag{52}$$

That is, we found the following family of special solution to (42):

$$\boxed{u_n(x,y) = A_n \sin(nx) \sinh(ny), \quad n \in \mathbb{N}.} \tag{53}$$

Since the Laplace equation with the boundary conditions in the regions R_1, R_2, R_3 (see figure on p. 41) satisfies the superposition principle, a general solution is of the form

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} A_n \sin(nx) \sinh(ny)}. \tag{54}$$

It remains to incorporate the boundary condition in the region R_4. We require

$$f(x) \overset{!}{=} u(x,\pi) = \sum_{n=1}^{\infty} A_n \sin(nx) \sinh(n\pi)$$

$$= \sum_{n=1}^{\infty} a_n \sin(nx) \quad \text{with} \quad a_n = A_n \sinh(n\pi). \tag{55}$$

But this is almost a Fourier series and we already encounter this case. As before we define the odd extension

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in [0,\pi], \\ -f(-x) & \text{if } x \in [-\pi, 0), \end{cases} \tag{56}$$

which we write as

$$\tilde{f}(x) = \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} c_n \sin(nx) \quad \text{with}$$

$$c_n = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} \sin(nx) \tilde{f}(x)\, dx$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\pi} \sin(nx) f(x)\, dx. \tag{57}$$

We conclude that

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(nx) \quad \text{with}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} \sin(nx) f(x)\, dx, \tag{58}$$

and hence,

$$\boxed{A_n = \frac{2}{\pi \sinh(\pi n)} \int_0^{\pi} \sin(nx) f(x)\, dx.} \tag{59}$$

In particular, we found that the solution to (42) subject to the boundary conditions in (43) is given by (54) with A_n in (59). This ends our investigation of the Laplace equation.