Chapter 1 : Ising model

1.0. Introduction

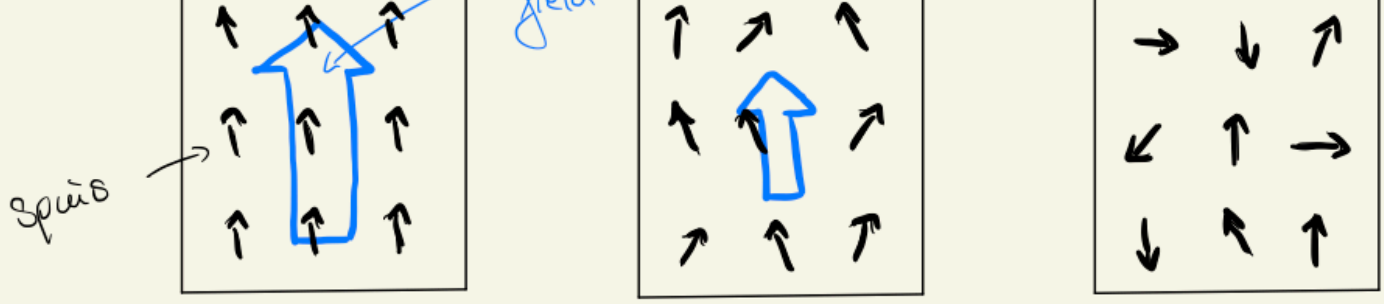
Already in the 19th century physicists came up with a phenomenological theory of heat and temperature called thermodynamics. This theory can be used for example to describe steam engines, the combustion engine in your car, or the refrigerator in your flat. Later in the 19th and 20th century after people had conjectured and later proven that all materials are made of atoms, people asked the question whether thermodynamics could be derived from the microscopic physics of atoms. From a physics point of view this relation has been understood within the framework of statistical mechanics.

In statistical mechanics one does not consider the individual motion of microscopic particles (atoms) but rather describes their typical behavior with a probability measure. How to guess the correct probability measure is one of the key insights from statistical mechanics. In this lecture we will not discuss how these measures can be justified. We will instead investigate their mathematical properties.

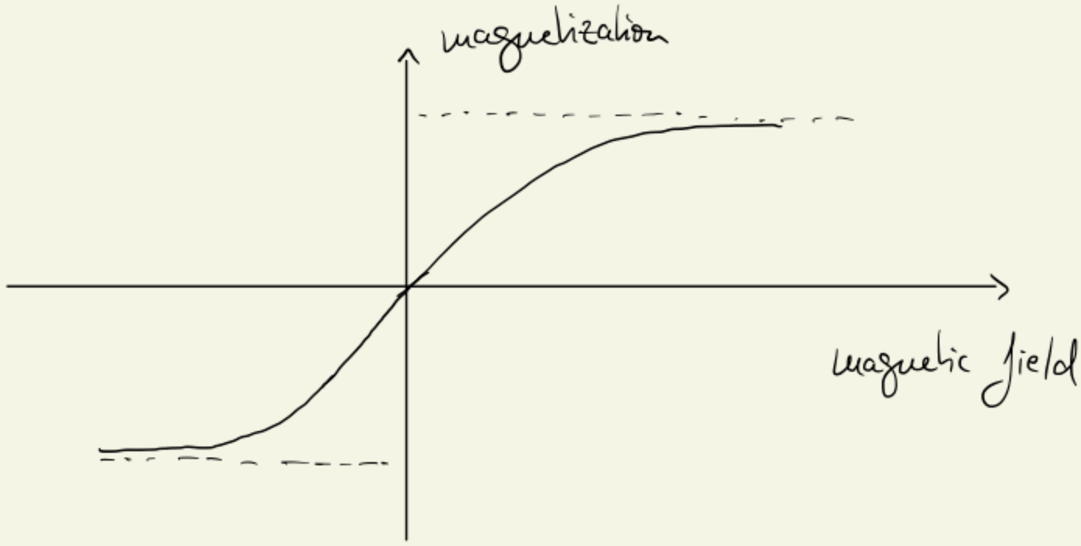
Much of the theory that we will discuss can be applied in various circumstances. To keep things concrete and not too astract we will always consider the case of a magnet. More precisely, we will consider a magnet described by the Ising model. Before we introduce this model, we quickly discuss the physical phenomenology that is described by it.

Let us consider a material consisting of microscopic magnets that interact with each other (the magnets tend to point in the same direction) as well as with an external magnetic field (the magnets tend to align with the magnetic field). In the following we call the microscopic magnets spins. There are two types of behavior of the spins described by the Ising model.

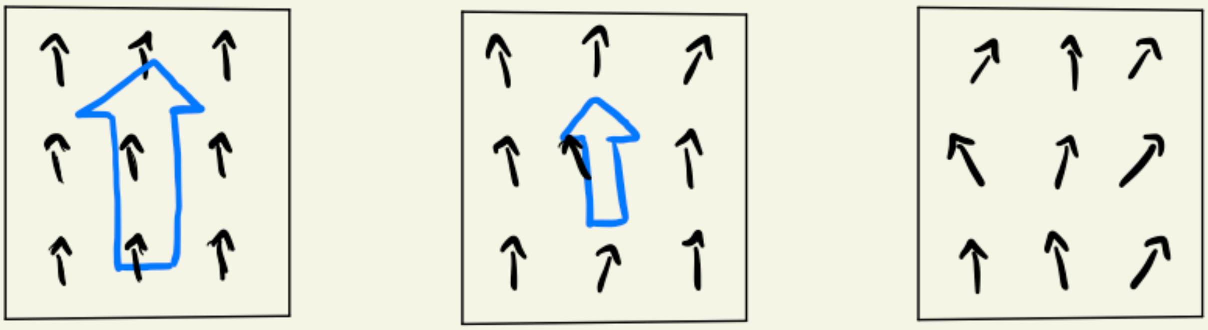
Paramagnetic behavior. In the first scenario the global order of the magnets is progressively lost as the external magnetic field goes to zero.



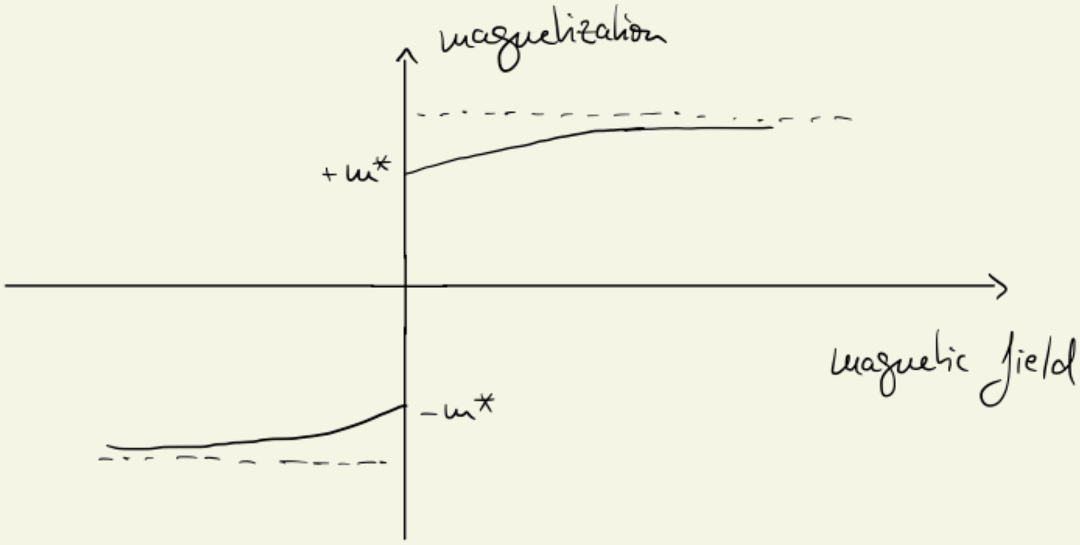
In this case the **magnetization** (vector sum of the spins) has the following dependence on the external magnetic field.



<u>Ferromagnetic behavior</u>: In the second scenario the global magnetization still decreases as a function of the external magnetic field. But the local interactions among the spins are strong enough for the material to maintain a globally magnetized state even after the external field has vanished.



A ferromagnet thus exhibits **spontaneous magnetization**, that is, global ordering of the spins in the absence of an external magnetic field. The value of $\pm m^*$ of the spontaneous magnetization depends on whether the external field approached zero from positive or negative values:



As the external field goes through zero, the magnetization is discontinuous (it jumps from $-m^*$ to $+m^*$).

Whether a material in nature displays paramagnetic or ferromagnetic behavior often depends on the **temperature** of the sample. If the temperature is large enough the sample will be paramagnetic and it is ferromagnetic for small temperatures. The point, where the system changes its behavior is usually unique and called the **Curie temperature**. The Ising model can be used to describe this behavior!

3.1. Finite volume Gibbs measures

In this section we define the Ising model on $\mathbb{Z}^d$ and establish some of its properties.

▪ Finite volume with free boundary condition.

The spins described by the Ising model are of length one and can only point in two different directions (± 1). In case of the model on $\mathbb{Z}^2$, for example, one can imagine that the spins are all parallel to the e_3-direction.

The configurations of the Ising model in a finite volume $\Lambda \Subset \mathbb{Z}^d$ with free boundary condition are the elements of the set

$$\Omega_\Lambda = \{-1, 1\}^\Lambda .$$

A configuration $\omega \in \Omega_\Lambda$ is thus of the form $\omega = (\omega_i)_{i \in \mathbb{Z}^d}$.

The basic random variable associated with the model is the **spin** at vertex $i \in \mathbb{Z}^d$, which is defined by

$$\sigma_i : \Omega_\Lambda \to \{-1, 1\},$$

$$\sigma_i(\omega) = \omega_i . \qquad (1)$$

We will often identify a finite set Λ with the graph that contains all edges formed by nearest-neighbor pairs of vertices of Λ. We denote the latter set of edges by

nearest neighbors

$$\mathcal{E}_\Lambda = \{ \{i,j\} \subset \Lambda : i \sim j \}. \qquad (2)$$

For each configuration $\omega \in \Omega_\Lambda$, we associate the energy, given by the Hamiltonian

$$\mathcal{H}^{\phi}_{\Lambda;\beta,h}(\omega) = -\beta \sum_{\{i,j\} \in \mathcal{E}_\Lambda} \sigma_i(\omega)\sigma_j(\omega) - h \sum_{i \in \Lambda} \sigma_i(\omega). \qquad (3)$$

Here $\beta > 0$ denotes the inverse temperature and $h \in \mathbb{R}$

is the external magnetic field. The superscript ϕ indicates that this model has **free boundary condition**, that is, spins in Λ do not interact with other spins located outside of Λ.

<u>Definition 3.1.</u> The Gibbs distribution of the Ising model with **free boundary condition**, at parameters β and h, is the probability distribution on Ω_Λ defined by

$$\mu^{\phi}_{\Lambda;\beta,h}(\omega) = \frac{1}{Z^{\phi}_{\Lambda;\beta,h}} \exp\left(-\mathcal{H}^{\phi}_{\Lambda;\beta,h}(\omega)\right). \tag{4}$$

The normalization constant

$$Z^{\phi}_{\Lambda;\beta,h} = \sum_{\omega \in \Omega_\Lambda} \exp\left(-\mathcal{H}^{\phi}_{\Lambda;\beta,h}(\omega)\right) \tag{5}$$

is called the **partition function** with free boundary condition.

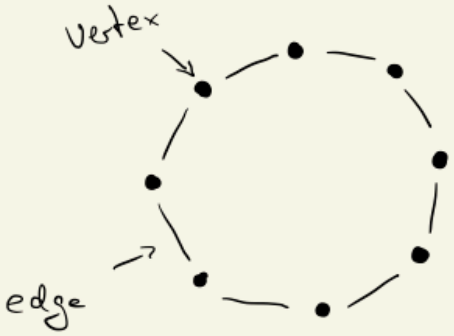
$\boxed{\cdot}$ Finite volume with periodic b.c.

One can also consider the Ising model on the torus $\mathbb{T}_n$ defined as follows: Its set of vertices is given by

$$V_n = \{0, \dots, n-1\}^d ,$$

and there is an edge between each pair of vertices $i = (i_1, \dots, i_d)$, $j = (j_1 \dots j_d)$ s.t. $\sum_{r=1}^{d} \left| (i_r - j_r) \bmod n \right| = 1$.

We denote by $\mathcal{E}_{V_n}^{per}$ the set of edges of $\mathbb{T}_n$.



One-dimensional torus with eight vertices.

Configurations of the model are now the elements $\{-1,1\}^{V_n}$ and have an energy given by

$$\mathcal{H}_{V_n; \beta, h}^{per}(\omega) = -\beta \sum_{\{i,j\} \in \mathcal{E}_{V_n}^{per}} \sigma_i(\omega) \sigma_j(\omega) - h \sum_{i \in V_n} \sigma_i(\omega). \qquad (6)$$

Definition 3.2. The Gibbs distribution of the Ising model in V_n with **periodic b.c.**, at parameters β and h, is the probability distribution on $\{-1,1\}^{V_n}$ defined by

$$\mu^{per}_{V_n;\beta,h}(\omega) = \frac{1}{Z^{per}_{V_n;\beta,h}} \exp\left(-\mathcal{H}^{per}_{V_n;\beta,h}(\omega)\right), \qquad (7)$$

where

$$Z^{per}_{V_n;\beta,h} = \sum_{\omega\in\Omega_{V_n}} \exp\left(-\mathcal{H}^{per}_{V_n;\beta,h}(\omega)\right) \qquad (8)$$

denotes the partition function with periodic b.c..

$\boxed{\cdot}$ Finite volume with configurations as b.c.

It will later be useful to consider the Ising model in a subset $\Lambda \Subset \mathbb{Z}^d$ with some frozen spins outside the set Λ.

Configurations of the Ising model on the infinite lattice $\mathbb{Z}^d$ are elements of the set

$$\Omega = \{-1, 1\}^{\mathbb{Z}^d}. \tag{9}$$

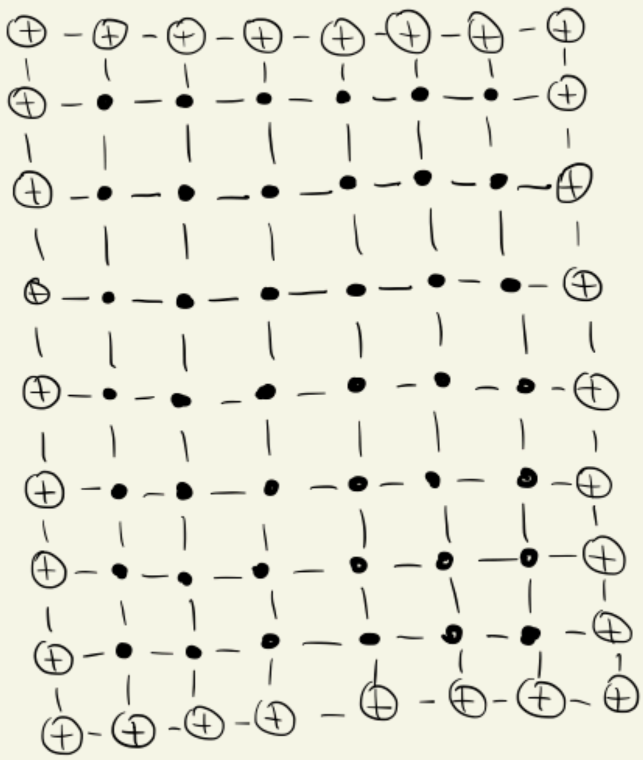
Let us fix a finite subset $\Lambda \Subset \mathbb{Z}^d$ and a configuration $\eta \in \Omega$. We define a **configuration of the Ising model in Λ with b.c. η** as an element of the finite set

$$\Omega_\Lambda^\eta = \{ \omega \in \Omega : \omega_i = \eta_i \ \ \forall\, i \notin \Lambda \}. \tag{10}$$

We also define

$$\mathcal{E}_\Lambda^b = \{ \{i,j\} \subset \mathbb{Z}^d : \{i,j\} \cap \Lambda \neq \phi, \ i \sim j \}. \tag{11}$$

Note that $\mathcal{E}_\Lambda^b$ differs from $\mathcal{E}_\Lambda$ by the addition of al the edges connecting vertices inside Λ to their neighbors outside Λ.



$\bullet$ = spins inside Λ.

$\oplus$ = neighboring frozen spins with value $+1$ outside of Λ.

The **energy** of a configuration $\omega \in \Omega_\Lambda^\eta$ is defined by

$$\mathcal{H}_{\Lambda;\beta,h}(\omega) = -\beta \sum_{\{i,j\} \in \mathcal{E}_\Lambda^b} \sigma_i(\omega)\sigma_j(\omega) - h \sum_{i \in \Lambda} \sigma_i(\omega). \tag{12}$$

<u>**Definition 3.3.**</u> The Gibbs distribution of the Ising model in Λ **with b.c. η**, at β and h, is the probability distribution on Ω_Λ^η defined by

$$\mu^{\eta}_{\Lambda;\beta,h}(\omega) = \frac{1}{Z^{\eta}_{\Lambda;\beta,h}} \exp\left(-\mathcal{H}_{\Lambda;\beta,h}(\omega)\right) \tag{13}$$

with partition function

$$Z^{\eta}_{\Lambda;\beta,h} = \sum_{\omega\in\Omega^{\eta}_{\Lambda}} \exp\left(-\mathcal{H}_{\Lambda;\beta,h}(\omega)\right). \tag{14}$$

The following boundary conditions will play a particularly important role in the following: the **+ b.c.** η^+, for which $\eta^+_i = +1$ $\forall i$, and the **− b.c.** η^-, defined by $\eta^-_i = -1$ $\forall i$. The corresponding Gibbs distributions will be denoted by $\mu^+_{\Lambda;\beta,h}$ and $\mu^-_{\Lambda;\beta,h}$. We will also write Ω^+_Λ, Ω^-_Λ for the corresponding sets of configurations.

We will also use the symbol # to denote generic b.c.s. E.g. $Z^{\#}_{\Lambda;\beta,h}$ may be $Z^{\phi}_{\Lambda;\beta,h}$, $Z^{per}_{\Lambda;\beta,h}$ or $Z^{\eta}_{\Lambda;\beta,h}$. Following

The custom in statistical physics, expectations of a function f w.r.t. a probability distribution μ will be denoted by $\langle f \rangle_\mu$. We will also use the notation

$$\langle f \rangle^{\#}_{\Lambda;\beta,h} = \sum_{\omega \in \Omega^{\#}_{\Lambda}} f(\omega)\, \mu^{\#}_{\Lambda;\beta,h}(\omega). \tag{15}$$

3.2. Thermodynamic limit, pressure and magnetization

Our ultimate goal is to understand the phase diagram of the Ising model. We will, for example, show that the magnetization "wants" to be non-zero for sufficiently small temperatures and $h=0$. In contrast, the magnetization always vanishes when the temperature is sufficiently large ($h=0$). To formulate and prove these kinds of statements it is very convenient to consider an infinite volume limit called the **thermodynamic limit**. This will be motivated in more detail later when we have the relevant definitions at hand.

3.2.1. Convergence of subsets

In this section we introduce a procedure to take the thermodynamic limit. It is appropriate for a proper description of thermodynamics and phase transitions.

To define the Ising model on the whole $\mathbb{Z}^d$ (i.e. in infinite volume), we will consider a limit along sequences of finite subsets $\Lambda_n \Subset \mathbb{Z}^d$, which **converge to** $\mathbb{Z}^d$, denoted $\Lambda_n \uparrow \mathbb{Z}^d$, in the sense that

1.) Λ_n is increasing: $\Lambda_n \subset \Lambda_{n+1}$,

2.) Λ_n invades $\mathbb{Z}^d$: $\bigcup_{n \geq 1} \Lambda_n = \mathbb{Z}^d$.

In order to control the influence of the boundary condition and of the shape of the box on thermodynamic quantities, it will be necessary to impose a further regularity property on $\{\Lambda_n\}_{n \geq 1}$.

We say that $\Lambda_n \uparrow \mathbb{Z}^d$ **converges to $\mathbb{Z}^d$ in the sense of von Hove**, which we denote by $\Lambda_n \Uparrow \mathbb{Z}^d$, iff

$$\lim_{n\to\infty} \frac{|\partial^{in}\Lambda_n|}{|\Lambda_n|} = 0, \tag{16}$$

where

$$\partial^{in}\Lambda = \{ i \in \Lambda : \exists j \notin \Lambda,\ i \sim j \}. \tag{17}$$

The simplest sequence to satisfy this condition is

$$B(n) = \{-n, \dots, n\}^d. \tag{18}$$

3.2.2. Pressure

The partition functions introduced above play a very important role in the theory, in particular, because they give rise to the pressure of the model.

The pressure has a clear physical interpretation and, as we will see later, it can be used to compute the magnetization.

Definition 3.4. The pressure in $\Lambda \Subset \mathbb{Z}^d$, with b.c. $\#$, is defined by

$$\psi_\Lambda^\#(\beta,h) = \frac{1}{|\Lambda|} \ln\left(Z^\#_{\Lambda;\beta,h} \right). \tag{19}$$

The following is an important observation.

Lemma 3.5. For all b.c.s $\#$, $(\beta,h) \mapsto \psi_\Lambda^\#(\beta,h)$ is convex.

Proof: We only consider the case of $\psi_\Lambda^\eta(\beta,h)$. The cases of the other b.c.s are similar. Using that

$\mathcal{H}_{\Lambda;\beta,h}$ is an affine function of (β,h) and Hölder's inequality, we see that $\left(\alpha\in[0,1]\right)$

$$Z^{\eta}_{\Lambda;\,\alpha\beta_1+(1-\alpha)\beta_2,\,\alpha h_1+(1-\alpha)h_2} = \sum_{\omega\in\Omega^{\eta}_{\Lambda}} \exp\left(-\alpha\,\mathcal{H}_{\Lambda;\beta_1,h_1}(\omega)-(1-\alpha)\,\mathcal{H}_{\Lambda;\beta_2,h_2}(\omega)\right)$$

$$\underset{\uparrow}{\leq} \left(\sum_{\omega\in\Omega^{\eta}_{\Lambda}}\exp\left(-\mathcal{H}_{\Lambda;\beta_1,h_1}(\omega)\right)\right)^{\alpha}\left(\sum_{\omega\in\Omega^{\eta}_{\Lambda}}\exp\left(-\mathcal{H}_{\Lambda;\beta_2,h_2}(\omega)\right)\right)^{1-\alpha}. \tag{20}$$

Hölder with
$p=\frac{1}{\alpha}$
$q=\frac{1}{1-\alpha}$

Since the logarithm is monotone increasing we thus have

$$\ln\left(Z^{\eta}_{\Lambda;\,\alpha\beta_1+(1-\alpha)\beta_2,\,\alpha h_1+(1-\alpha)h_2}\right)\leq \alpha\ln\left(Z^{\eta}_{\Lambda;\beta_1,h_1}\right) + (1-\alpha)\ln\left(Z^{\eta}_{\Lambda;\beta_2,h_2}\right), \tag{21}$$

which implies the claim. $\blacksquare$

The finite volume pressure $\Psi_\Lambda^\#$ depends on Λ and the b.c. used. However, when Λ is large in a way that $|\Lambda| \gg |\partial^{in}\Lambda|$ holds, the b.c. and the shape of Λ only provide negligible corrections: there exists a function $\Psi(\beta,h)$ s.t.

$$\Psi_\Lambda^\#(\beta,h) = \Psi(\beta,h) + O\left(\frac{|\partial^{in}\Lambda|}{|\Lambda|}\right). \tag{22}$$

$\Psi(\beta,h)$ is called the infinite volume pressure (or just pressure) and provides a better candidate for the corresponding thermodynamic potential (it does not depend on "details" of the system as e.g. its shape).

Theorem 3.6. In the thermodynamic limit, the **pressure**

$$\Psi(\beta,h) = \lim_{\Lambda \Uparrow \mathbb{Z}^d} \Psi_\Lambda^\#(\beta,h) \tag{23}$$

is well defined. In particular, it is independent of the sequence $\Lambda \uparrow \mathbb{Z}^d$ and of the b.c. Moreover, $\mathcal{F}$ is convex (on $\mathbb{R}_+ \times \mathbb{R}$) and is even as a function of h.

Proof:

Existence of the limit: We first consider the case of free b.c. and argue in two steps. In the first step we show existence of the limit

$$\lim_{n\to\infty} \mathcal{F}^{\phi}_{D_n}(\beta, h), \tag{24}$$

where $D_n = \{1, 2, \dots, 2^n\}^d$. After that, we extend the convergence to any sequence $\Lambda_n \uparrow \mathbb{Z}^d$. Since (β, h) is fixed, it will be omitted from the notation.

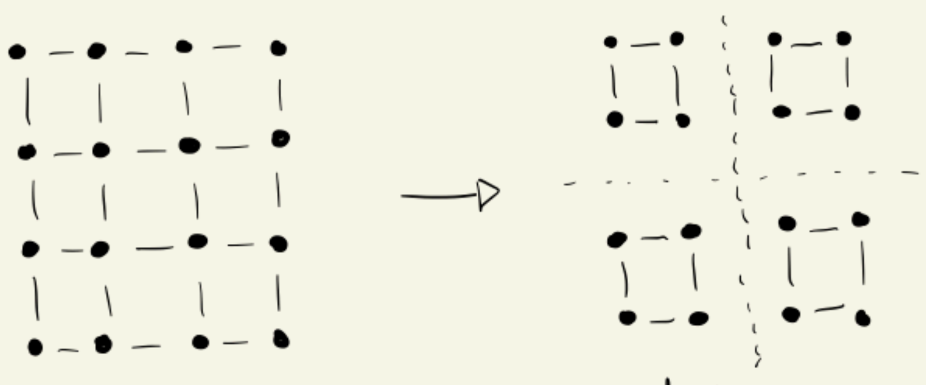


fig. 1

D_2 $\qquad$ 2^d translates of D_1 $(d=2)$

We will show that $\Psi^{\phi}_{D_{n+1}}$ is close to $\Psi^{\phi}_{D_n}$. Let us decompose D_{n+1} into 2^d disjoint translates of D_n (see fig. 1), denoted by $D_n^{(1)}, \dots, D_n^{(2^d)}$. The energy of ω in D_{n+1} can be written as

$$\mathcal{H}^{\phi}_{D_{n+1}} = \sum_{i=1}^{2^d} \mathcal{H}^{\phi}_{D_n^{(i)}} + R_n \,, \tag{25}$$

where R_n denotes the energy related to the interaction between pairs of spins that belong to different $D_n^{(i)}$. To obtain a rough bound for R_n, it suffices to note that (a) there are 2^d translates of D_n, (b) each of these has no more than $2d$ neighbors (see fig. 1), (c) one face of B_n contains $(2^n)^{d-1}$ points. Hence,

$$\begin{aligned} |R_n| &\leq \beta\, 2^d\, 2d\, (2^n)^{d-1} = \beta d\, 2^{d+1} (2^n)^{d-1} \\ &= 4\beta d\, (2^{n+1})^{d-1}. \end{aligned} \tag{26}$$

Note that this is a boundary contribution because it is not proportional to $(2^n)^d$ $(= \text{volume})$. The Hamiltonian in (25) is therefore bounded from below by

$$\mathcal{H}^{\phi}_{D_{n+1}} \geq \sum_{i=1}^{2^d} \mathcal{H}^{\phi}_{D_n} - 4\beta d \left(2^{n+1}\right)^{d-1}, \tag{27}$$

and the partition function satisfies

$$Z^{\phi}_{D_{n+1}} \leq \exp\left(4\beta d\, 2^{(n+1)(d-1)}\right) \underbrace{\sum_{\omega \in \Omega_{D_{n+1}}} \prod_{i=1}^{2^d} \exp\left(-\mathcal{H}^{\phi}_{D_n^{(i)}}(\omega)\right)}$$

$$= \prod_{i=1}^{2^d} \sum_{\omega \in \Omega_{D_n^{(i)}}} \exp\left(-\mathcal{H}^{\phi}_{D_n^{(i)}}(\omega)\right)$$

$$= \exp\left(4\beta d\, 2^{(n+1)(d-1)}\right) \left(Z^{\phi}_{D_n}\right)^{2^d}, \tag{28}$$

where we used $Z^{\phi}_{D_n^{(i)}} = Z^{\phi}_{D_n}$ (all the cubes are the same).

Eq. (27) equally holds when "$\geq$" is replaced by "$\leq$" and "$-$" is replaced by "$+$". Hence

$$\exp\left(-4\beta d\ 2^{(n+1)(d-1)}\right)\left(Z^{\phi}_{D_n}\right)^{2^d} \leq Z^{\phi}_{D_{n+1}} \tag{29}$$

$$\leq \exp\left(4\beta d\ 2^{(n+1)(d-1)}\right)\left(Z^{\phi}_{D_n}\right)^{2^d}.$$

We take the logarithm on both sides of these inequalities, divide by $|D_{n+1}| = 2^{d(n+1)}$ and find

$$\left|\Psi^{\phi}_{D_{n+1}} - \Psi^{\phi}_{D_n}\right| \leq 4\beta d\ 2^{-(n+1)}, \tag{30}$$

as well as $(n<m)$

$$\left|\Psi^{\phi}_{D_m} - \Psi^{\phi}_{D_n}\right| \leq 4\beta d \sum_{k=n+1}^{m} 2^{-(k+1)} = 8\beta d\left(2^{-n} - 2^{-m}\right). \tag{31}$$

We conclude that $\Psi^{\phi}_{D_n}$ is a Cauchy-sequence, and hence the limit $\lim_{n\to\infty} \Psi^{\phi}_{D_n} = \Psi$ exists.

Next, we consider an arbitrary sequence of sets with $\Lambda_n \Uparrow \mathbb{Z}^d$. We fix $k \in \mathbb{N}$ and consider a partition of $\mathbb{Z}^d$

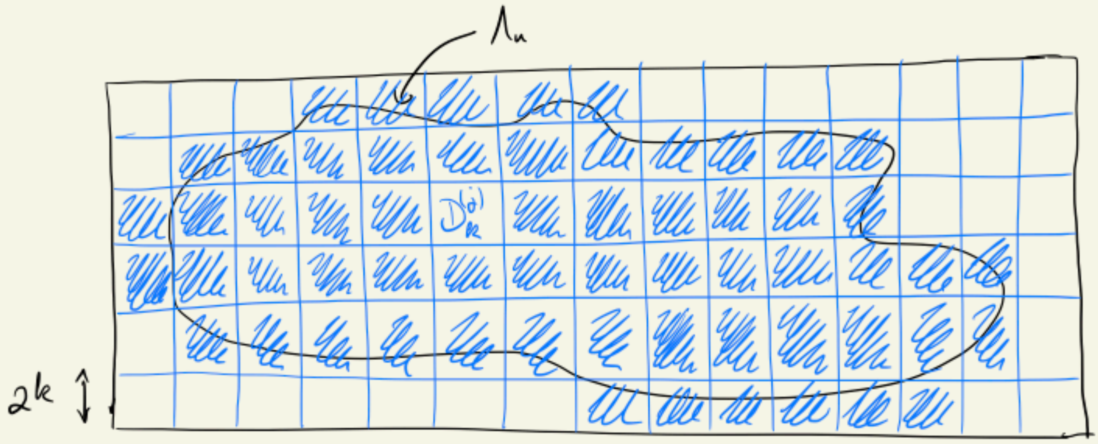


fig. 2

into adjacent disjoint translated of D_k, see fig. 2. For each n, we consider a minimal covering of Λ_n by elements $D_k^{(j)}$ of the partition and define

$$[\Lambda_n] = \bigcup_j D_k^{(j)}. \tag{32}$$

To compare Ψ to $\Psi_{\Lambda_n}^{\phi}$, we estimate

$$|\Psi_{\Lambda_n}^{\phi} - \Psi| \leq |\Psi_{\Lambda_n}^{\phi} - \Psi_{[\Lambda_n]}^{\phi}| + |\Psi_{[\Lambda_n]}^{\phi} - \Psi_{D_k}^{\phi}| + |\Psi_{D_k}^{\phi} - \Psi|. \tag{33}$$

Fix $\varepsilon > 0$. There exists $k_0 \in \mathbb{N}$ (depending on ε) s.t. $|\Psi_{D_k}^{\phi} - \Psi| \leq \varepsilon/3$ as long as $k \geq k_0$ (we know $\Psi_{D_k}^{\phi} \to \Psi$).

To obtain a bound for the second term on the r.h.s. of (33) we write

$$\mathcal{H}^{\phi}_{[\Lambda_n]} = \sum_j \mathcal{H}^{\phi}_{D_k^{(j)}} + W_n, \tag{34}$$

and note that

$$|W_n| \le \beta \underbrace{\frac{|[\Lambda_n]|}{\underbrace{|D_k|}_{=(2^k)^d}}}_{\substack{\#\text{ of cubes} \\ \text{(see fig. 2)}}} \overbrace{2d}^{\#\text{ neighbors of one cube has}} \underbrace{2^{k(d-1)}}_{\substack{\#\text{ of points of one face} \\ \text{of } D_k}} = 2\beta d\, 2^{-k}\, |[\Lambda_n]|. \tag{35}$$

Hence,

$$\left| \Psi^{\phi}_{[\Lambda_n]} - \Psi^{\phi}_{D_k} \right| \le \beta d\, 2^{-(k-1)} \tag{36}$$

and there exists $k_1 \in \mathbb{N}$ s.t. the r.h.s. is smaller than $\varepsilon/3$ provided $k \ge k_1$. It remains to consider the first term on the r.h.s. of (33).

To that end, we define $\Delta_n = [\Lambda_n] \setminus \Lambda_n$ and observe that

$$\left| \mathcal{H}^{\phi}_{\Lambda_n} - \mathcal{H}^{\phi}_{[\Lambda_n]} \right| \leq \left(2d\beta + |h| \right) |\Delta_n| , \tag{37}$$

which implies

$$Z^{\phi}_{[\Lambda_n]} = \sum_{\omega \in \Omega_{[\Lambda_n]}} \exp\left(-\mathcal{H}^{\phi}_{[\Lambda_n]}(\omega)\right) \lesseqgtr \sum_{\omega \in \Omega_{\Lambda_n}} \exp\left(-\mathcal{H}^{\phi}_{\Lambda_n}(\omega)\right)$$

$$\times \sum_{\omega' \in \Omega_{\Delta_n}} \exp\left(\pm \left(2d\beta + |h|\right) |\Delta_n| \right)$$

$$= \exp\left[\left(\pm 2d\beta \pm |h| \pm \ln(2) \right) |\Delta_n| \right] Z^{\phi}_{\Lambda_n} . \tag{38}$$

We note that $|\Delta_n| \leq |\partial^{in} \Lambda_n| \, |D_R|$. When we put this together with (28), we find

$$\left| \ln Z^{\phi}_{[\Lambda_n]} - \ln Z^{\phi}_{\Lambda_n} \right| \leq |\partial^{in} \Lambda_n| \, |D_R| \left(2d\beta + |h| + \ln(2) \right). \tag{39}$$

To lift this to a bound for the related pressures, we use $\mathcal{F}^{\phi}_{\Lambda_n} \leq 2d\beta + |h| + \ln(2)$ as well as

$$1 \leq \frac{|[\Lambda_n]|}{|\Lambda_n|} = \frac{|\Lambda_n| + \Delta_n}{|\Lambda_n|} \leq 1 + \frac{|\partial^{in}\Lambda_n|\, D_k}{|\Lambda_n|} \tag{40}$$

and find

$$\left|\mathcal{F}^{\phi}_{\Lambda_n} - \mathcal{F}^{\phi}_{[\Lambda_n]}\right| \leq 2\,\frac{|\partial^{in}\Lambda_n|\,|D_k|}{|\Lambda_n|}\left(2d\beta + |h| + \ln(2)\right). \tag{41}$$

For fixed $k \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ s.t. $n \geq n_0$ implies that the r.h.s. is smaller than $\varepsilon/3$. When we combine these considerations (and choose first $k \geq \max\{k_0, k_1\}$ and afterwards $n \geq n_0(k_1, k_2)$), we find

$$\left|\mathcal{F}^{\phi}_{\Lambda_n} - \mathcal{F}\right| \leq \varepsilon. \tag{42}$$

This proves the claimed convergence of the pressure for free b.c..

⊡ **Independence of the b.c.:** That all b.c.s. lead to the same limit follows when we combine the bounds

$$|\mathcal{H}_\Lambda^{\#} - \mathcal{H}_\Lambda^{\phi}| \le 2d\beta\,|\partial^{in}\Lambda|, \tag{43}$$

where $\# = \eta$ or $\# = per$, and arguments that appear in the first part of the proof.

- **Convexity:** From Lemma 3.5 we know that $(\beta,h) \mapsto \psi_\Lambda^{\#}(\beta,h)$ is convex. To obtain the convexity of $\psi(\beta,h)$, we have to use that the pointwise limit of a sequence of convex functions is convex. You will prove this statement in the exercise class.

- **Symmetry:** This is a direct consequence of the symmetry of the pressure in finite volume. You will prove the symmetry in finite volume in the exercise class.

3.2.3. Magnetization

Another quantity of general importance is the **magnetization density** in $\Lambda \Subset \mathbb{Z}^d$, which is the random variable

$$m_\Lambda(\omega) = \frac{1}{|\Lambda|} M_\Lambda(\omega) = \frac{1}{|\Lambda|} \sum_{i \in \Lambda} \sigma_i(\omega). \qquad (44)$$

The quantity M_Λ is called the **total magnetization.** We also define

$$m^\#_\Lambda(\beta,h) = \langle m_\Lambda \rangle^\#_{\Lambda;\beta,h}. \qquad (45)$$

We have the following important identity (please check!)

$$m^\#_\Lambda(\beta,h) = \frac{\partial \psi^\#_\Lambda}{\partial h}(\beta,h). \qquad (46)$$

Remark: As you will show in the exercises, the **cumulant generating function** $f(t) = \ln\left(\langle e^{tM_\Lambda}\rangle^\#_{\Lambda;\beta,h}\right)$ of the random variable M_Λ (In case you never heard of cumulant generating functions, please read about it in Friedli, Velenik, Appendix B.8.3. or check out the relevant Wikipedia page) can be written as

$$f(t) = |\Lambda| \left(\psi_\Lambda^{\#}(\beta, h+t) - \psi_\Lambda^{\#}(\beta, h) \right). \tag{47}$$

This, in particular, implies that the r-th cumulant of M_Λ is given by

$$C_r(M_\Lambda) = |\Lambda| \frac{\partial^r \psi_\Lambda^{\#}}{\partial h^r} (\beta, h). \tag{48}$$

As explained in Friedli, Velenik, Appendix B.8.3., the cumulant generating function of a random variable contains all information about its distribution. In view of the central role played by the magnetization in the characterization of phase transitions, the pressure should therefore also hold important information about the occurance of phase transitions. At first sight this may seem surprising because the pressure is defined via the partition function, which is just a normalization factor appearing in the Gibbs distribution.

An important question to be asked is, whether (46) also holds in the thermodynamic limit. To answer this question one needs to overcome two difficulties:

(1) One needs to show that $\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{\partial \psi_\Lambda^{\#}}{\partial h}(\beta, h)$ exists and does not depend on the b.c..

(2) We need to ask the question whether the thermodynamic limit and differentiation w.r.t. h can be interchanged:

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{\partial \psi_\Lambda^{\#}}{\partial h} \overset{?}{=} \frac{\partial}{\partial h} \lim_{\Lambda \uparrow \mathbb{Z}^d} = \frac{\partial \psi}{\partial h}. \tag{48}$$

These issues are related to the differentiability of the (infinite volume) pressure ψ, which is again related to the occurance of phase transitions.

We will give a full answer to these questions in Section 3.7. For the moment, we will use the convexity of the pressure to give a partial answer.

Item 1 of Theorem B.12 in Friedli, Velenik implies that the one-sided derivatives

$$\frac{\partial \psi}{\partial h^-}(\beta,h) = \lim_{h'\uparrow h} \frac{\psi(\beta,h-h')-\psi(\beta,h)}{(-h')} \quad \text{and}$$

$$\frac{\partial \psi}{\partial h^+}(\beta,h) = \lim_{h'\uparrow h} \frac{\psi(\beta,h+h')-\psi(\beta,h)}{h'} \tag{50}$$

of the map $h \mapsto \psi(\beta,h)$ exist everywhere. Moreover, $h \mapsto \frac{\partial \psi}{\partial h^-}$ is left-continuous and $h \mapsto \frac{\partial \psi}{\partial h^+}$ is right-continuous. The pressure is differentiable w.r.t. h iff $\frac{\partial \psi}{\partial h^-} = \frac{\partial \psi}{\partial h^+}$ holds. Because of this, we introduce the set

$$\begin{aligned} B_\beta &= \{ h\in\mathbb{R} : \psi(\beta,\cdot) \text{ is not differentiable at } h \} \\ &= \{ h\in\mathbb{R} : \frac{\partial \psi}{\partial h^-}(\beta,h) \neq \frac{\partial \psi}{\partial h^+}(\beta,h) \}. \end{aligned} \tag{51}$$

From Theorem B.12 we know that B_β is at most countable. On the complement of this set we can answer the question raised on p. 33.

Corollary 3.7. For all $h \notin \mathcal{B}_\beta$, the **average magnetization density**

$$m(\beta,h) = \lim_{\Lambda \Uparrow \mathbb{Z}^d} m_\Lambda^{\#}(\beta,h) \tag{52}$$

is well defined, independent of the sequence $\Lambda \Uparrow \mathbb{Z}^d$ and the b.c., and satisfies

$$m(\beta,h) = \frac{\partial \psi}{\partial h}(\beta,h). \tag{53}$$

Moreover, the function $h \mapsto m(\beta,h)$ is non-decreasing on $\mathbb{R} \setminus \mathcal{B}_\beta$ and is continuous at every $h \notin \mathcal{B}_\beta$. It is, however, discontinuous at each $h \in \mathcal{B}_\beta$: for any $h \in \mathcal{B}_\beta$,

$$\lim_{h' \downarrow h} m(\beta,h') = \frac{\partial \psi}{\partial h^+}(\beta,h), \qquad \lim_{h' \uparrow h} m(\beta,h') = \frac{\partial \psi}{\partial h^-}(\beta,h). \tag{54}$$

In particular, the **spontaneous magnetization**

$$m^*(\beta) = \lim_{h \downarrow 0} m(\beta,h) \tag{55}$$

is always well defined.

Proof: Assume that $h \notin B_\beta$. Then we have

$$\frac{\partial f}{\partial h}(\beta,h) = \frac{\partial}{\partial h} \lim_{\Lambda \Uparrow \mathbb{Z}^d} f_\Lambda^{\#}(\beta,h)$$

$$\underset{\substack{\nearrow \\ \text{item 7 of Thm. B12}}}{=} \lim_{\Lambda \Uparrow \mathbb{Z}^d} \frac{\partial f_\Lambda^{\#}}{\partial h}(\beta,h) \underset{\substack{\nearrow \\ (46)}}{=} \lim_{\Lambda \Uparrow \mathbb{Z}^d} m_\Lambda^{\#}(\beta,h). \tag{56}$$

This proves the existence of the thermodynamic limit of the magnetization density, that it depends neither on the sequence of volumes nor on the b.c., and (53).

The monotonicity and continuity of $h \mapsto m(\beta,h)$ on $\mathbb{R} \setminus B_\beta$ follow from (53) and items no. 4 and 5 of Thm. B12.

Choose $h \in \mathcal{B}_\beta$. Since $\mathcal{B}_\beta$ is at most countable, there exists a sequence $\{h_k\}_{k\geq 1}$ s.t. $h_k \downarrow h$. From (53) we know that $\frac{\partial \psi}{\partial h^+}(\beta, h_k) = m(\beta, h_k)$ for all $k \geq 1$. The claim (54) now follows from item no. 5 of Thm. B.12. This proves the claim. $\blacksquare$

3.2.4. A first definition of phase transition

From the above Corollary we know that $h \mapsto m(\beta,h)$ is discontinuous iff $h \mapsto \psi(\beta,h)$ is not differentiable. This motivates the following definition.

Definition 3.P. The pressure ψ exhibits a **first order phase transition** at (β,h) if $h \mapsto \psi(\beta,h)$ fails to be differentiable at that point.

Later we will introduce another notion of first order phase transition, which is more probabilistic. Determining whether a phase transitions occur or not, and at which values of the parameters, is one of the main objectives of this Section.

3.3. One-dimensional Ising model

Before we come to the general case, we discuss the one-dimensional Ising model. It has the advantage of being exactly solvable, and therefore serves as a good example.

Theorem 3.8. For all $\beta \geq 0$ and all $h \in \mathbb{R}$, the pressure $\psi(\beta, h)$ of the one-dimensional Ising model is given by

$$\psi(\beta,h) = \ln\left(e^{\beta} \cosh(h) + \sqrt{e^{2\beta}\cosh^2(h) - 2\sinh(2\beta)} \right). \quad (57)$$

Before we prove the theorem, let us discuss its implications.

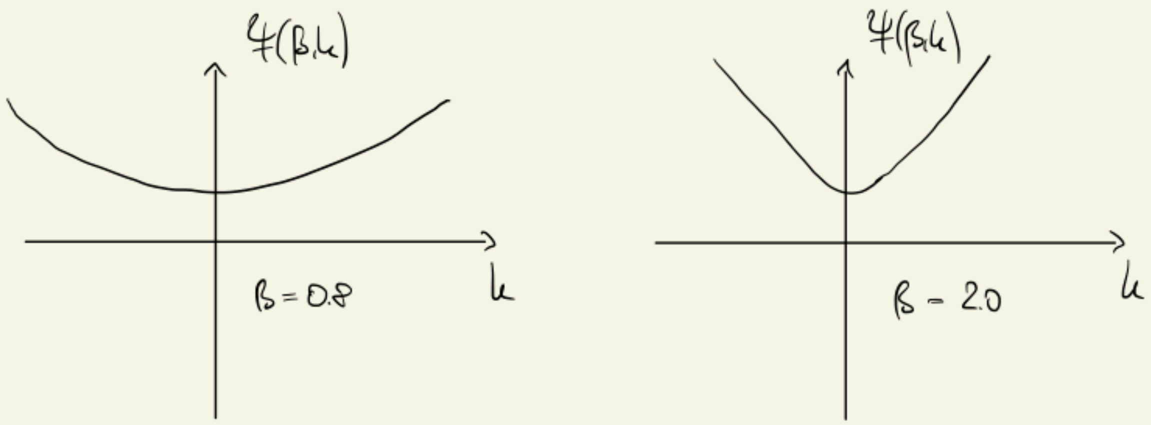


fig. 3

The function in (57) is everywhere differentiable in h (it is actually real analytic), for all $\beta > 0$. Accordingly, $\mathcal{S}_\beta = \emptyset$ when $d=1$. We have $m(\beta,h) = \frac{\partial \Psi}{\partial h}(\beta,h)$ for all $h \in \mathbb{R}$. Since $h \mapsto \Psi(\beta,h)$ is real analytic, the same is true for $m(\beta,h)$. Hence, $m^*(\beta) = \lim_{h \downarrow 0} m(\beta,h) = m(\beta,0)$. From $\Psi(\beta,h) = \Psi(\beta,-h)$, we conclude that $\frac{\partial \Psi}{\partial h}(\beta,0) = 0$. That is, the spontaneous magnetization is zero:

$$m^*(\beta) = 0 \quad \forall \beta > 0. \tag{58}$$

In particular, the 1d Ising model displays **paramagnetic** behavior and no phase transition occurs in the system.

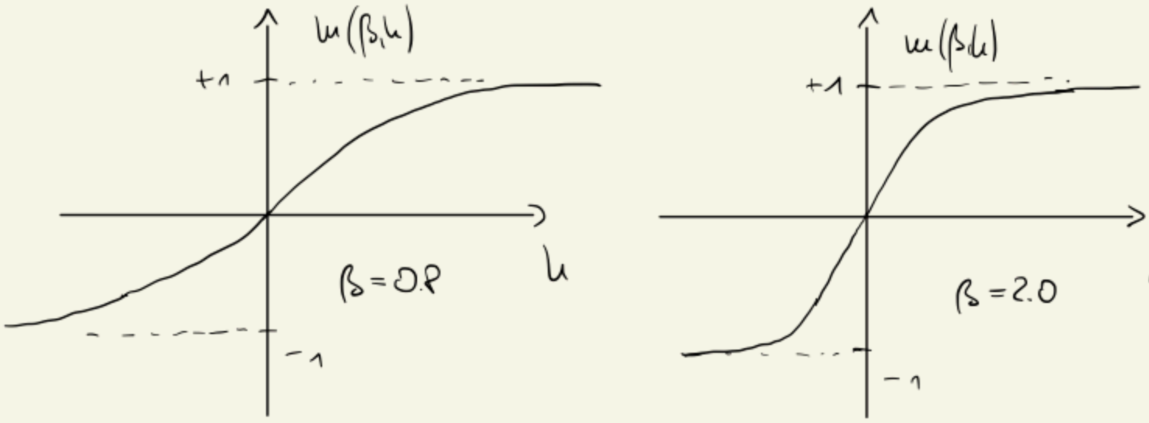


fig. 4

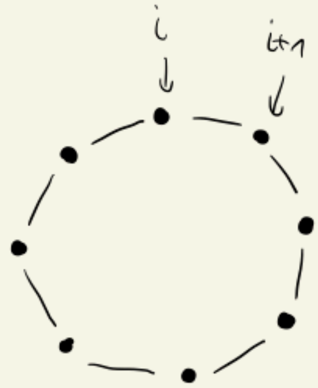
Only in the limit $\beta \to 0$ does $f(\beta,h)$ become non-differentiable at $h=0$. You will prove this in the exercise group.

Proof of Thm. 3.9. We have shown in Thm. 3.6. that the pressure is indep. of the choice of b.c. and of the sequence of volumes $\Lambda \uparrow \mathbb{Z}^d$. The most convenient choice to work with is the torus T_n ($V_n = \{0, \dots, n-1\}$ with periodic b.c.). In this case the partition function can be written as

$$Z^{per}_{V_n;\beta,h} = \sum_{\omega \in \Omega_{V_n}} \exp\left(-\mathcal{H}^{per}_{V_n;\beta,h}(\omega)\right) =$$

$$= \sum_{\omega\in\Omega_{V_n}} \exp\Big(\beta \sum_{\{i,j\}\in\mathcal{E}^{per}_{V_n}} \sigma_i(\omega)\sigma_j(\omega) + h\sum_{i\in V_n}\sigma_i(\omega)\Big)$$

$$= \sum_{\omega_0=\pm1}\cdots\sum_{\omega_{n-1}=\pm1} \prod_{i=0}^{n-1} \underbrace{\exp\big(\beta\omega_i\omega_{i+1}+h\omega_i\big)}_{=A_{\omega_i,\omega_{i+1}}},$$



where $A_{+1,+1}=e^{\beta+h}$, $A_{+1,-1}=e^{-\beta+h}$, $A_{-1,+1}=e^{-\beta-h}$, and $A_{-1,-1}=e^{\beta-h}$. In matrix notation, this reads

$$A = \begin{pmatrix} e^{\beta+h} & e^{-\beta+h} \\ e^{-\beta-h} & e^{\beta-h} \end{pmatrix}. \tag{59}$$

The matrix A is called the **transfer matrix**. An important observation is that $Z^{per}_{V_n;\beta,h}$ can be written as the trace of the n-th power of A:

$$Z^{per}_{V_n;\beta,h} = \operatorname{tr}[A^n]. \tag{60}$$

A straightforward computation shows that the eigenvalues of A are given by

$$\lambda_\pm = e^\beta \cosh(h) \pm \sqrt{e^{2\beta}\cosh^2(h) - 2\sinh(2\beta)}. \qquad (61)$$

We evaluate the trace in the eigenbasis of A and find

$$Z^{per}_{V_n;\beta,h} = \operatorname{tr}\left[A^n\right] = \lambda_+^n + \lambda_-^n. \qquad (62)$$

Since $\lambda_+ > \lambda_-$, we have

$$\begin{aligned} f(\beta,h) &= \lim_{n\to\infty} \underbrace{\frac{1}{|V_n|}}_{=\frac{1}{n}} \ln\Big(\underbrace{Z^{per}_{V_n;\beta,h}}_{\lambda_+^n + \lambda_-^n \,=\, \lambda_+^n\left(1+\left(\frac{\lambda_-}{\lambda_+}\right)^n\right)}\Big) \\ &= \lim_{n\to\infty} \frac{1}{n}\Big[n\ln(\lambda_+) + \ln\Big(1+\underbrace{\left(\frac{\lambda_-}{\lambda_+}\right)^n}_{\xrightarrow[n\to\infty]{} 0}\Big)\Big] \\ &= \ln(\lambda_+). \end{aligned} \qquad (63)$$

This proves the claim of our theorem. $\blacksquare$

As we have seen, an explicit formula for the pressure yields very useful information about the system. Unfortunately, computing the pressure becomes much more difficult, if at all possible, in higher dimentions. The only other known result in this direction concerns the two-dimentional Ising model with $h=0$. In that case, Onsager determined, in a celebrated work (1944), the explicit expression for the pressure:

$$\psi(\beta,0) = \ln(2) + \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi} \ln\Big(\cosh^2(2\beta) - \sinh(2\beta)\big(\cos(\theta_1) + \cos(\theta_2)\big)\Big)\, d\theta_1 d\theta_2 . \tag{64}$$

We will not discuss this formula here, and rather focus on the development of more general techniques that allow us to study the Ising model for all $d \geq 1$.

Remark: In the next section, we will introduce infinite volume Gibbs states (in some sense extentions of finite volume Gibbs distributions in infinite volume). We will see that smoothness of the pressure guarantees uniqueness of infinite-volume Gibbs states.

3.4. Infinite Volume Gibbs states

The pressure provides information about the existence of phase transitions and allows us to compute the expectation of the magnetization density (actually all cumulants of the magnet. dens. if we control arbitrary derivatives of the pressure). We may, however, be interested in other observables as e.g. the correlation between two (or more) far apart spins. To compute such kind of correlation functions we need to understand the behavior of the Gibbs distribution in large volumes. One way of doing this is to define infinite volume Gibbs measures by taking some sequence $\Lambda_n \uparrow \mathbb{Z}^d$ and by considering the accumulation points of sequences of the type $\{\mu^{\eta_n}_{\Lambda_n;\beta,h}\}_{n=1}^{\infty}$. This is possible and will be done later. In this chapter we follow a route that is mathematically less involved and more hands-on.

We will define infinite volume states as linear functionals acting on the set of functions that depend only on a finite number of spins.

<u>Definition 3.11.</u> A function $f : \Omega \to \mathbb{R}$ is **local** if there exists $\Delta \Subset \mathbb{Z}^d$ s.t. $f(\omega) = f(\omega')$ as soon as ω and ω' coincide on Δ. The smallest such set Δ is called the **support** of f and denoted by $\mathrm{supp}(f)$.

<u>**Examples:**</u> The functions $\sigma_0(\omega)$ and the magnetization density in a finite volume $\Lambda \Subset \mathbb{Z}^d$, $m_\Lambda(\omega) = \frac{1}{|\Lambda|}\sum_{i\in\Lambda} \sigma_i(\omega)$, are local functions with $\mathrm{supp}(\sigma_0) = \{0\}$ and $\mathrm{supp}(m_\Lambda) = \Lambda$.

<u>Remark 3.12.</u> We will make the following abuse of notation: If $f : \Omega \to \mathbb{R}$ is a local function with $\Delta \supset \mathrm{supp}(f)$, then, for any $\omega' \in \Omega_\Delta$, $f(\omega')$ is defined as $f(\omega)$, where $\omega \in \Omega$ denotes any configuration with $\omega_i = \omega'_i \ \forall\, i \in \Delta$.

Definition 3.13. An **infinite-volume state** (or simply a **state**) is a mapping associating to each local function f a real number $\langle f \rangle$ that satisfies:

-) **Normalization**: $\langle 1 \rangle = 1$.
-) **Positivity**: If $f \geq 0$, then $\langle f \rangle \geq 0$.
-) **Linearity**: $\langle f + \lambda g \rangle = \langle f \rangle + \lambda \langle g \rangle \quad \forall \lambda \in \mathbb{R}$. (65)

The number $\langle f \rangle$ is called the **average** (or **expectation**) **of** f **in the state** $\langle \cdot \rangle$.

Definition 3.14. Let $\Lambda_n \uparrow \mathbb{Z}^d$ and $\{\#_n\}_{n=1}^{\infty}$ be a sequence of boundary conditions. The sequence of Gibbs distributions $\{\mu_{\Lambda_n; \beta, h}^{\#_n}\}_{n=1}^{\infty}$ is said to **converge to the state** $\langle \cdot \rangle$ iff

$$\lim_{n \to \infty} \langle f \rangle_{\Lambda_n; \beta, h}^{\#_n} = \langle f \rangle, \tag{66}$$

for every local fct. f. The state $\langle \cdot \rangle$ is then called a **Gibbs** state (at (β, h)).

In the following we simply write $\langle \cdot \rangle = \lim_{n\to\infty} \langle \cdot \rangle^{\#_n}_{\Lambda_n;\beta,h}$ to indicate convergence.

<u>**Remark 3.15.**</u> If you are familiar with functional analysis you will probably have noticed that, using the Riesz–Markov representation theorem, the average $\langle f \rangle$ of a local function f in a state $\langle \cdot \rangle$ can always be seen as the expectation of f under some probability measure μ on $\{+1,-1\}^{\mathbb{Z}^d}$:

$$\langle f \rangle = \int f \, d\mu. \tag{67}$$

We are mostly interested in states that can be constructed as limits of finite-volume Gibbs distributions. As we will see later, in this case, the measure μ is a suitable weak limit of finite volume Gibbs distributions.

Our states are defined on the infinite lattice and so it is natural to single out those that are translation-invariant. To that end, we first define the **translation** by $j \in \mathbb{Z}^d$

$$\Theta_j : \mathbb{Z}^d \to \mathbb{Z}^d \quad \text{by}$$

$$\Theta_j i = i + j. \tag{68}$$

Translations can naturally act on configurations: if $\omega \in \Omega$, then

$$(\Theta_j \omega)_i = \omega_{i-j}. \tag{69}$$

Definition 3.16. A state $\langle \circ \rangle$ is **translation invariant** if $\langle f \circ \Theta_j \rangle = \langle f \rangle$ holds for every local function f and for all $j \in \mathbb{Z}^d$.

The first important question is: can we construct Gibbs states for the Ising model with parameters (β, h)? The following theorem shows that the boundary conditions η^+ and η^- can be used to construct two states that will play a central role subsequently.

Theorem 3.17. Let $\beta \geq 0$ and $h \in \mathbb{R}$. Along any sequence $\Lambda_n \uparrow \mathbb{Z}^d$, the finite volume Gibbs distributions with + or − boundary conditions converge to infinite volume Gibbs states:

$$\langle \cdot \rangle^{\pm}_{\beta,h} = \lim_{n\to\infty} \langle \cdot \rangle^{\pm}_{\Lambda_n;\beta,h} . \tag{70}$$

The states $\langle \cdot \rangle^{\pm}_{\beta,h}$ do not depend on the sequence $\{\Lambda_n\}_{n=1}^{\infty}$ and are both translation-invariant.

The proof of the theorem will be given later after we have introduced some important tools.

Remark 3.18. Note that there is no claim in the theorem that $\langle \cdot \rangle^{+}_{\beta,h}$ and $\langle \cdot \rangle^{-}_{\beta,h}$ are distinct. Determining the values

β and h for which this is the case will be one of our main tasks in the remainder of this section.

3.5. Two families of local functions

The construction of Gibbs states requires proving the existence of the limit $\lim_{n\to\infty} \langle f \rangle^{\eta_n}_{\Lambda_n;\beta,h}$ for each local function. The following lemma allows us to restrict this question to two subclasses of functions. They are especially well suited for the use of the correlation inequalities introduced in the next section.

For $A \Subset \mathbb{Z}^d$, we define

$$\sigma_A = \prod_{j\in A} \sigma_j \quad \text{and} \quad n_A = \prod_{j\in A} n_j \,, \tag{71}$$

where

$$n_j = \tfrac{1}{2}\left(1+\sigma_j\right) \tag{72}$$

is the **occupation variable** at j.

Lemma 3.19. Let f be a local function. There exist real coefficients $\{\hat{f}_A\}_{A \subset \mathrm{supp}(f)}$ and $\{\tilde{f}_A\}_{A \subset \mathrm{supp}(f)}$ s.t. both of the following representations hold:

$$f = \sum_{A \subset \mathrm{supp}(f)} \hat{f}_A \sigma_A , \quad f = \sum_{A \subset \mathrm{supp}(f)} \tilde{f}_A \eta_A . \tag{73}$$

Proof: We start by proving the following orthogonality relation: for all $B \Subset \mathbb{Z}^d$ and all configurations $\omega, \tilde{\omega}$,

$$2^{-|B|} \sum_{A \subset B} \sigma_A(\tilde{\omega}) \sigma_A(\omega) = \underbrace{\mathbb{1}}_{\text{obvious meaning ☺}} \left(\omega_i = \tilde{\omega}_i , \ \forall i \in B \right). \tag{74}$$

Let us first assume that $\tilde{\omega}_i = \omega_i \ \forall i \in B$. In this case we have

$$\sigma_A(\tilde{\omega}) \sigma_A(\omega) = \prod_{i \in A} \underbrace{\sigma_i(\omega) \sigma_i(\omega')}_{= \sigma_i^2(\omega) = 1} = 1 , \tag{75}$$

as well as

$$\sum_{A \subset B} \sigma_A(\tilde{\omega}) \sigma_A(\omega) = \sum_{A \subset B} 1 = 2^{|B|}, \tag{76}$$

and hence (74) holds. Assume now that there exists $i \in B$ s.t. $\omega_i \neq \tilde{\omega}_i$. Then

$$\sum_{A \subset B} \sigma_A(\omega) \sigma_A(\tilde{\omega}) = \sum_{A \subset B \setminus \{i\}} \left(\sigma_A(\omega) \sigma_A(\tilde{\omega}) + \sigma_{A \cup \{i\}}(\omega) \sigma_{A \cup \{i\}}(\tilde{\omega}) \right)$$

$$= \sum_{A \subset B \setminus \{i\}} \left(\sigma_A(\omega) \sigma_A(\tilde{\omega}) + \omega_i \tilde{\omega}_i \sigma_A(\omega) \sigma_A(\tilde{\omega}) \right)$$

$$= \sum_{A \subset B \setminus \{i\}} \left(\sigma_A(\omega) \sigma_A(\tilde{\omega}) \left(1 + \underbrace{\omega_i \tilde{\omega}_i}_{= -1} \right) \right) = 0. \tag{77}$$

This proves (74).

Next, we apply (74) with $B = \mathrm{supp}(f)$ and find

$$f(\omega) = \sum_{\omega' \in \Omega_{\mathrm{supp}(f)}} f(\omega') \, \mathbb{1}\left(\omega_i = \omega'_i, \ \forall i \in \mathrm{supp}(f) \right)$$

$$= \sum_{\omega' \in \mathrm{supp}(f)} f(\omega') \, 2^{-|\mathrm{supp}(f)|} \sum_{A \subset \mathrm{supp}(f)} \sigma_A(\omega) \sigma_A(\omega')$$

$$= \sum_{A \subset \mathrm{supp}(f)} \underbrace{\left[2^{-|\mathrm{supp}(f)|} \sum_{\omega' \in \mathrm{supp}(f)} f(\omega') \sigma_A(\omega') \right]}_{\hat{f}_A} \sigma_A(\omega). \qquad (78)$$

This proves the first identity (73) with $\hat{f}_A$ in (78).

The second identity in (73) follows from the first because

$$\sigma_A = \prod_{i \in A} (2n_i - 1). \qquad (79)$$

This proves the claim. ∎

Thanks to the above lemma we can, when aiming to show convergence of $\{\langle f\rangle_{\Lambda_n;\beta,h}^{\eta_n}\}_{n\geq 1}$ for all local functions f, restrict attention to $\{\langle \sigma_A\rangle_{\Lambda_n;\beta,h}^{\eta_n}\}_{n\geq 1}$ or $\{\langle n_A\rangle_{\Lambda_n;\beta,h}^{\eta_n}\}_{n\geq 1}$ for all finite $A\subset\mathbb{Z}^d$. This problem will be solved with the help of the correlation inequalities we introduce in the next section.

3.6. Correlation inequalities

There exist many correlation inequalities for the Ising model. We will restrict attention to the most prominent ones: the GKS and the FKG inequalities.

3.6.1. GKS Inequalities

Motivation: The Ising model favors the alignement of spins (the energy of aligned spins is lower than that of non-aligned spins). In case of the model with $+-$ b.c. and $h \geq 0$ we would therefore expect a nonnegative magnetization in finite volume. It therefore seems reasonable to expect that we have

$$\langle \sigma_i \rangle^+_{\Lambda;\beta,h} \geq 0 \tag{80}$$

for all $i \in \Lambda$. Similarly, knowing that the spin at vertex j takes the value $+1$ should not decrease the probability of observing a $+$ spin at another given vertex i. That is, one would expect

$$\underbrace{\mu^+_{\Lambda;\beta,h}(\sigma_i = 1 \mid \sigma_j = 1)}_{\text{conditional probability}} \geq \mu^+_{\Lambda;\beta,h}(\sigma_i = 1), \tag{81}$$

$$\boxed{P(A \mid B) = \frac{P(A \cap B)}{P(B)}}$$

$\Leftrightarrow$

$$\mu^+_{\Lambda;\beta,h}(\sigma_i = 1, \sigma_j = 1) \geq \mu^+_{\Lambda;\beta,h}(\sigma_i = 1)\, \mu^+_{\Lambda;\beta,h}(\sigma_j = 1).$$

Using $\mathbb{1}(\sigma_i = 1) = \frac{1}{2}(\sigma_i + 1)$, we can write this also as

$$\langle \sigma_i \sigma_j \rangle^+_{\Lambda,\beta,h} \geq \langle \sigma_i \rangle^+_{\Lambda,\beta,h} \langle \sigma_j \rangle^+_{\Lambda,\beta,h} . \tag{82}$$

The r.h.s. minus the l.h.s. is called the covariance of the two random variables σ_i and σ_j. In our case this quantity is nonnegative and we say σ_i and σ_j are **positively correlated** under $\mu^+_{\Lambda,\beta,h}$. Eqs. (80) and (82) are true, and will be particular instances of the GKS inequalities (named after Griffith, Kelly and Sherman) stated below.

We state them in a more general setting. For $\Lambda \in \mathbb{Z}^d$, let $\underline{K} = \{K_C\}_{C \subset \Lambda}$ be a family of real numbers, called **coupling constants**. Consider the following probability distribution on Ω_Λ:

$$\nu_{\Lambda,\underline{K}} = \frac{1}{Z_{\Lambda,\underline{K}}} \exp\Big(\sum_{C \subset \Lambda} K_C \, \omega_C \Big), \tag{83}$$

where $\omega_C = \prod_{i\in C} \omega_i$ and $Z_{\Lambda;\underline{K}}$ is the associated partition function. The Gibbs distributions $\mu^+_{\Lambda;\beta,h}$, $\mu^\phi_{\Lambda;\beta,h}$ and $\mu^{per}_{\Lambda;\beta,h}$ can all be written in this form with $K_C \geq 0 \ \forall C \subset \Lambda$ if $h \geq 0$. In the case of $\mu^+_{\Lambda;\beta,h}$ we e.g. need to make the choice

$$K_C = \begin{cases} h + \beta \# \{ j \notin \Lambda : i \sim j \} & \text{if } C = \{i\} \subset \Lambda, \\ \beta & \text{if } C = \{i,j\} \subset \Lambda \ i \sim j, \\ 0 & \text{otherwise.} \end{cases} \tag{84}$$

Theorem 3.20 (GKS inequalities) Let $\underline{K} = \{K_C\}_{C\subset\Lambda}$ be s.t. $K_C \geq 0 \ \forall C \subset \Lambda$. Then, for any $A, B \subset \Lambda$,

$$\langle \sigma_A \rangle_{\Lambda;\underline{K}} \geq 0, \tag{85}$$

$$\langle \sigma_A \sigma_B \rangle_{\Lambda;\underline{K}} \geq \langle \sigma_A \rangle_{\Lambda;\underline{K}} \langle \sigma_B \rangle_{\Lambda;\underline{K}}. \tag{86}$$

Proof: When we expand the exponential

$$\exp\left(K_C \omega_C\right) = \sum_{n_C \in \mathbb{N}_0} \frac{1}{n_C!} K_C^{n_C} \omega_C^{n_C} , \tag{87}$$

we can write

$$Z_{\Lambda;\underline{K}} \langle \sigma_A \rangle_{\Lambda;\underline{K}} = \sum_{\omega \in \Omega_\Lambda} \omega_A \prod_{C \subset \Lambda} \underbrace{\exp\left(K_C \omega_C\right)}_{\sum_{n_C \in \mathbb{N}_0} \frac{1}{n_C!} K_C^{n_C} \omega_C^{n_C}}$$

$$= \sum_{\substack{\{n_C\}_{C \subset \Lambda} \\ n_C \in \mathbb{N}_0}} \left[\prod_{C \subset \Lambda} \frac{K_C^{n_C}}{n_C!} \right] \sum_{\omega \in \Omega_\Lambda} \omega_A \left[\prod_{C \subset \Lambda} \omega_C^{n_C} \right] . \tag{88}$$

We rewrite

$$\omega_A \prod_{C \subset \Lambda} \omega_C^{n_C} = \prod_{i \in \Lambda} \omega_i^{m_i} , \tag{89}$$

where

$$m_i = \mathbb{1}(i \in A) + \sum_{C \subset \Lambda : i \in C} n_C . \tag{90}$$

Since

$$\sum_{\omega_i = \pm 1} \omega_i^{m_i} = \begin{cases} 2 & \text{if } m_i \text{ is even} \\ 0 & \text{if } m_i \text{ is odd,} \end{cases} \tag{91}$$

it follows that

$$\sum_{\omega \in \Omega_\Lambda} \prod_{i \in \Lambda} \omega_i^{m_i} = \prod_{i \in \Lambda} \sum_{\omega_i = \pm 1} \omega_i^{m_i} \geq 0. \tag{92}$$

Putting (88), (89) and (92) together, we find

$$Z_{\Lambda; \underline{K}} \langle \sigma_A \rangle_{\Lambda; \underline{K}} = \sum_{\substack{\{n_c\}_{c \subset \Lambda} \\ n_c \in \mathbb{N}}} \left[\prod_{c \subset \Lambda} \frac{K_c^{n_c}}{n_c!} \right] \underbrace{\sum_{\omega \in \Omega_\Lambda} \prod_{i \in \Lambda} \omega_i^{m_i}}_{= \prod_{i \in \Lambda} \sum_{\omega_i = \pm 1} \omega_i^{m_i} \geq 0}$$

$$\geq 0. \tag{93}$$

This proves (85). It remains to prove (86).

To that end, we duplicate the system. That is, we consider the product distribution $\nu_{\Lambda;\underline{K}} \otimes \nu_{\Lambda;\underline{K}}$ on $\Omega_\Lambda \times \Omega_\Lambda$ defined by

$$\nu_{\Lambda;\underline{K}} \otimes \nu_{\Lambda;\underline{K}} (\omega, \omega') = \nu_{\Lambda;\underline{K}}(\omega)\, \nu_{\Lambda;\underline{K}}(\omega'). \tag{84}$$

If we define $\sigma_i(\omega,\omega') = \omega_i$ and $\sigma_i'(\omega,\omega') = \omega_i'$, then

$$\langle \sigma_A \sigma_B \rangle_{\Lambda;\underline{K}} - \langle \sigma_A \rangle_{\Lambda;\underline{K}} \langle \sigma_B \rangle_{\Lambda;\underline{K}} = \langle \sigma_A (\sigma_B - \sigma_B') \rangle. \tag{85}$$

The problem of proving (86) is thus equivalent to proving that

$$0 \overset{!}{\leq} \left(Z_{\Lambda;\underline{K}}\right)^2 \langle \sigma_A (\sigma_B - \sigma_B') \rangle_{\nu_{\Lambda;\underline{K}} \otimes \nu_{\Lambda;\underline{K}}}$$

$$= \sum_{\omega,\omega' \in \Omega_\Lambda} \omega_A (\omega_B - \omega_B') \prod_{C \subset \Lambda} \exp\left(K_C (\omega_C + \omega_C') \right) = (*) \tag{86}$$

holds. Next, we introduce the variable $\omega_i'' = \omega_i \omega_i' = \frac{\omega_i'}{\omega_i}$ and write

$$(*) = \sum_{\omega,\omega' \in \Omega_\Lambda} \omega_A \omega_B (1-\omega''_B) \prod_{C \subset \Lambda} \exp\left(K_C(1+\omega''_C)\omega_C\right) \qquad (97)$$

$$= 2 \sum_{\omega'' \in \Omega_\Lambda} (1-\omega''_B) \sum_{\omega \in \Omega_\Lambda} \omega_A \omega_B \prod_{C \subset \Lambda} \exp\left(K_C(1+\omega''_C)\omega_C\right).$$

We have $1-\omega''_B \geq 0$. If we treat the sum over $\omega \in \Omega_\Lambda$ (for fixed ω'') as the one in (88), but this time with the coupling constants $K_C(1+\omega''_C) \geq 0$, (86) follows.

$\blacksquare$

3.6.2. FGK inequality

The FGK inequality (named after Fortuin, Kasteleyn and Ginibre) states that increasing events (we explain in a second what that is) are positively correlated.

The total order of the set $\{-1, 1\}$ induces a **partial order** on Ω. We say that $\omega \leq \omega'$ holds iff $\omega_i \leq \omega'_i$

for all $i \in \mathbb{Z}^d$. An event $E \subset \Omega$ is **increasing** if $\omega \in E$ and $\omega \le \omega'$ implies $\omega' \in E$. If E and F are both increasing events depending on the spins inside Λ, then again, by the ferromagnetic nature of the model, one can expect that the occurence of an increasing event enhances the probability of another increasing event. That is, if F has positive probability, then

$$\mu^{+}_{\Lambda;\beta,h}(E \mid F) \geq \mu^{+}_{\Lambda;\beta,h}(E)$$

$$\iff \mu^{+}_{\Lambda;\beta,h}(E \cap F) \geq \mu^{+}_{\Lambda;\beta,h}(E)\, \mu^{+}_{\Lambda;\beta,h}(F). \qquad (SP)$$

As in the case of the GKS inequality, we will state and prove the FKG in a more general setting that we introduce next.

Let $\underline{J} = \{J_{ij}\}_{\{i,j\} \in \mathcal{E}^b_\Lambda}$ and $\underline{h} = \{h_i\}_{i \in \Lambda}$ be two collections of real numbers. We assume that $J_{ij} \geq 0 \ \forall\, i,j$ but $h_i \in \mathbb{R}$.

For $\omega \in \Omega_\Lambda^\eta$, we denote

$$\mathcal{H}_{\Lambda;\underline{J},\underline{h}}(\omega) = -\sum_{\{i,j\}\in\mathcal{E}_\Lambda^b} J_{ij}\,\sigma_i(\omega)\,\sigma_j(\omega) - \sum_{i\in\Lambda} h_i\sigma_i(\omega), \qquad (99)$$

and we use the obvious notation for the related Gibbs distribution and expectation value of observables. A function $f:\Omega\to\mathbb{R}$ is called **non-decreasing** iff $\omega\leq\omega'$ implies $f(\omega)\leq f(\omega')$. Our goal is to prove that

$$\langle fg\rangle^\eta_{\Lambda;\underline{J},\underline{h}} \geq \langle f\rangle^\eta_{\Lambda;\underline{J},\underline{h}}\,\langle g\rangle^\eta_{\Lambda;\underline{J},\underline{h}} \qquad (100)$$

holds for any $\Lambda\Subset\mathbb{Z}^d$ and any non-decreasing functions f,g. The statement we will prove is a bit more general that (100) and contains it as a special case.

For two configurations $\omega,\omega'\in\Omega_\Lambda$ we define

$$\omega \wedge \omega' = \left(\omega_i \wedge \omega_i'\right)_{i\in\Lambda}$$
$$\omega \vee \omega' = \left(\omega_i \vee \omega_i'\right)_{i\in\Lambda}. \qquad (101)$$

$$\boxed{\begin{aligned} x \wedge y &\overset{\text{def.}}{=} \min\{x,y\} \\ x \vee y &= \max\{x,y\} \end{aligned}}$$

Theorem 8.21.(a) Let $\mu = \bigotimes_{i\in\Lambda} \mu_i$ be a product measure on Ω_Λ. Let $f_1, \dots, f_4 : \Omega_\Lambda \to \mathbb{R}$ be nonnegative functions on Ω_Λ such that

$$f_1(\omega) f_2(\omega) \leq f_3(\omega \wedge \omega') f_4(\omega \vee \omega'), \quad \forall \omega, \omega' \in \Omega_\Lambda. \quad (102)$$

Then

$$\langle f_1 \rangle_\mu \langle f_2 \rangle_\mu \leq \langle f_3 \rangle_\mu \langle f_4 \rangle_\mu. \qquad (103)$$

Before we give the proof of the above theorem, let us explain why it implies (100). W.l.o.g. we can assume that f and g are both nonnegative and depend only on spins inside Λ. For $i \in \Lambda$, $s \in \{\pm 1\}$, let

$$\mu_i(s) = \exp\Big(hs + s \sum_{j \notin \Lambda,\, j \sim i} J_{ij}\, \eta_j \Big). \tag{104}$$

We have

$$\mu = \bigotimes_{i \in \Lambda} \mu_i$$

$$\langle f \rangle^{\eta}_{\Lambda; \underline{J}, \underline{h}} = \sum_{\omega \in \Omega_\Lambda} f(\omega)\, \rho(\omega)\, \mu(\omega) = \langle f \rho \rangle_\mu \,, \tag{105}$$

where

$$\rho(\omega) = \frac{\exp\Big(\sum_{\{i,j\} \in \mathcal{E}^b_\Lambda} J_{ij}\, \omega_i \omega_j \Big)}{Z^{\eta}_{\Lambda; \underline{J}, \underline{h}}} . \tag{106}$$

Choose $f_1 = \rho f$, $f_2 = \rho g$, $f_3 = \rho$, $f_4 = \rho f g$. If (102) holds for this choice, then (103) holds, and (100) is proved. To check (102) we need verify that

$$\rho(\omega)\, \rho(\omega') \leq \rho(\omega \vee \omega')\, \rho(\omega \wedge \omega'). \tag{107}$$

To see this, we first show that

$$\omega_i\omega_j + \omega_i'\omega_j' \le (\omega_i \vee \omega_i')(\omega_j \vee \omega_j') + (\omega_i \wedge \omega_i')(\omega_j \wedge \omega_j') \quad (108)$$

holds. The inequality is obvious if both terms on the r.h.s. are equal to 1. Let us therefore assume that at least one of them equals -1. This cannot happen if both $\omega_i \neq \omega_i'$ and $\omega_j \neq \omega_j'$. W.l.o.g. we can thus assume that $\omega_i = \omega_i'$. In that case the r.h.s. equals

$$\omega_i \left[(\omega_j \vee \omega_j') + (\omega_j \wedge \omega_j') \right] = \omega_i (\omega_j + \omega_j')$$

$$= \omega_i\omega_j + \omega_i'\omega_j', \qquad (109)$$

which proves (108). To prove (107) it remains to note that

$$p(\omega)p(\omega') = \frac{\exp\left(\sum_{\{i,j\}\in\mathcal{E}^b_\Lambda} J_{ij}\,(\omega_i\omega_j + \omega_i'\omega_j')\right)}{\left(Z^\eta_{\Lambda;\underline{J},\underline{h}}\right)^2} \underset{(108)}{\le}$$

$$\leq \frac{\exp\left(\sum_{\{i,j\}\in\mathcal{E}^b_\Lambda} J_{ij}\left((\omega_i \vee \omega_i')(\omega_j \vee \omega_j') + (\omega_i \wedge \omega_i')(\omega_j \wedge \omega_j')\right)\right)}{\left(Z^{\eta}_{\Lambda;\underline{J},\underline{h}}\right)^2}$$

$$= \rho(\omega \vee \omega')\, \rho(\omega \wedge \omega'). \tag{110}$$

Proof of Thm. 3.21.(a) For fixed $i \in \Lambda$, any config. $\omega \in \Omega_\Lambda$ can be identified with $(\tilde{\omega}, \omega_i)$, where $\tilde{\omega} \in \Omega_{\Lambda \setminus \{i\}}$. We will show that

$$f_1(\omega) f_2(\omega') \leq f_3(\omega \wedge \omega') f_4(\omega \vee \omega') \tag{110}$$

implies

$$\tilde{f}_1(\tilde{\omega}) \tilde{f}_2(\tilde{\omega}') \leq \tilde{f}_3(\tilde{\omega} \wedge \tilde{\omega}') \tilde{f}_4(\tilde{\omega} \vee \tilde{\omega}'), \tag{111}$$

where $\tilde{f}_k(\tilde{\omega}) = \langle f_k(\tilde{\omega}, \cdot) \rangle_{\mu_i} = \sum_{v=\pm 1} f_k(\tilde{\omega}, v)\, \mu_i(v)$. When we use this step $|\Lambda|$-times, we obtain (103).

The l.h.s. of (111) can be written as

$$\left\langle f_1(\tilde{\omega}, u) f_2(\tilde{\omega}', v) \right\rangle_{\mu_i \otimes \mu_i} = \left\langle \mathbb{1}_{\{u=v\}} f_1(\tilde{\omega}, u) f_2(\tilde{\omega}, v) \right\rangle \tag{112}$$

$$+ \left\langle \mathbb{1}_{\{u<v\}} \Big(f_1(\tilde{\omega}, u) f_2(\tilde{\omega}', v) + f_1(\tilde{\omega}, v) f_2(\tilde{\omega}', u) \Big) \right\rangle_{\mu_i \otimes \mu_i}$$

and the r.h.s. of (111) can be written as

$$\left\langle f_3(\tilde{\omega} \wedge \tilde{\omega}', u) f_4(\tilde{\omega} \vee \tilde{\omega}', v) \right\rangle_{\mu_i \otimes \mu_i}$$

$$= \left\langle \mathbb{1}_{\{u=v\}} f_3(\tilde{\omega} \wedge \tilde{\omega}', u) f_4(\tilde{\omega} \vee \tilde{\omega}', v) \right\rangle_{\mu_i \otimes \mu_i} \tag{113}$$

$$+ \left\langle \mathbb{1}_{\{u<v\}} \Big(f_3(\tilde{\omega} \wedge \tilde{\omega}, u) f_4(\tilde{\omega} \vee \tilde{\omega}', v) + f_3(\tilde{\omega} \wedge \tilde{\omega}', v) f_4(\tilde{\omega} \vee \tilde{\omega}', u) \Big) \right\rangle_{\mu_i \otimes \mu_i}$$

We thus obtain

$$\tilde{f}_3(\tilde{\omega} \wedge \tilde{\omega}') \tilde{f}_4(\tilde{\omega} \vee \tilde{\omega}') - \tilde{f}_1(\tilde{\omega}) \tilde{f}_2(\tilde{\omega}')$$

$$= \left\langle \mathbb{1}_{\{u=v\}} \Big(f_3(\tilde{\omega} \wedge \tilde{\omega}', u) f_4(\tilde{\omega} \vee \tilde{\omega}', v) - f_1(\tilde{\omega}, u) f_2(\tilde{\omega}', v) \Big) \right\rangle_{\mu_i \otimes \mu_i}$$

$$+ \left\langle \mathbb{1}_{\{u<v\}} \left(C + D - A - B \right) \right\rangle_{\mu_i \otimes \mu_i} , \tag{114}$$

where

$$A = f_1(\tilde{\omega}, u) f_2(\tilde{\omega}', v) , \quad B = f_1(\tilde{\omega}, v) f_2(\tilde{\omega}', u),$$

$$C = f_3(\tilde{\omega} \wedge \tilde{\omega}', u) f_4(\tilde{\omega} \vee \tilde{\omega}', v),$$

$$D = f_3(\tilde{\omega} \wedge \tilde{\omega}', v) f_4(\tilde{\omega} \vee \tilde{\omega}', u). \tag{115}$$

An application of (110) shows that the first term on the r.h.s. of (114) is nonnegative. Eq. (111) therefore follows if we can show that $A + B \leq C + D$.

We first note that (110) implies $A \leq C$, $B \leq D$, and

$$\begin{aligned} AB &= f_1(\tilde{\omega}, u) f_2(\tilde{\omega}', v) f_1(\tilde{\omega}, v) f_2(\tilde{\omega}', u) \\ &\leq f_3(\tilde{\omega} \wedge \tilde{\omega}', u) f_4(\tilde{\omega} \vee \tilde{\omega}', u) f_3(\tilde{\omega} \wedge \tilde{\omega}', v) f_4(\tilde{\omega} \vee \tilde{\omega}', v) \\ &= CD. \end{aligned} \tag{116}$$

We conclude that if $C=0$, then $A=B=0$ and the inequality $A+B \leq C+D$ is obvious. If $C \neq 0$, the inequality follows from

$$\frac{C+D-A-B}{C} \geq 1 + \frac{AB}{C^2} - \frac{A+B}{C} = \Big(1-\underbrace{\frac{A}{C}}_{\leq 1}\Big)\Big(1-\underbrace{\frac{B}{C}}_{\leq 1}\Big) \geq 0. \qquad (117)$$

use $\frac{D}{C} \geq \frac{AB}{C^2} \Leftrightarrow CD \geq AB$ (116)

$A \leq C$ $\quad$ $B \leq C$

This proves Thm. 3.2.1. $\blacksquare$

For later reference we state (100) as a theorem.

Theorem 3.21 (b) (FKG Inequality) Let $\underline{J} = \{J_{ij}\}_{ij\in\mathcal{E}^b_\Lambda}$ be a collection of nonnegative real numbers and let $\underline{h} = \{h_i\}_{i\in\Lambda}$ be a collection of arbitrary real numbers. Let $\Lambda \Subset \mathbb{Z}^d$ and $\#$ be some arbitrary b.c.. Then, for any pair of nondecreasing functions f and g, we have

$$\langle fg \rangle^{\#}_{\Lambda;\underline{J},\underline{h}} \geq \langle f \rangle^{\#}_{\Lambda;\underline{J},\underline{h}} \langle g \rangle^{\#}_{\Lambda;\underline{J},\underline{h}} . \tag{118}$$

Remark: • Eq. (98) follows from the theorem by taking $J_{ij} = \beta$ if $i \sim j$, $J_{ij} = 0$ otherwise; $h_i = h$ $\forall i \in \Lambda$, and $f = \mathbb{1}_E$, $g = \mathbb{1}_F$. It is important to note that $\mathbb{1}_E$ is a nondecreasing function if E is an increasing event.

• The proof of Theorem 3.21 (a) does not rely on the fact that the spins take their values in $\{\pm 1\}$; it actually holds for arbitrary real-valued spins.

• Theorem 3.22 (b) can be seen as a natural extension of the following elementary result: let $f, g : \mathbb{R} \to \mathbb{R}$ be nondecreasing and let μ be a probability measure on $\mathbb{R}$. We then have

$$\langle fg \rangle_\mu \geq \langle f \rangle_\mu \langle g \rangle_\mu . \tag{119}$$

To see this, we write

$$\langle fg \rangle_\mu - \langle f \rangle_\mu \langle g \rangle_\mu = \frac{1}{2} \int_{\mathbb{R}} \big(f(x)-f(y)\big)\big(g(x)-g(y)\big)\, \mu(dx)\, \mu(dy), \tag{120}$$

and observe that $f(x)-f(y)$ and $g(x)-g(y)$ always have the same sign.

3.6.3 Consequences

The first consequence of the correlation inequalities is the following lemma, which will be applied to study the thermodynamic limit.

Lemma 3.22. *Let f be a nondecreasing function and $\Lambda_1 \subset \Lambda_2 \Subset \mathbb{Z}^d$. Then, for any $\beta \geq 0$ and $h \in \mathbb{R}$,*

$$\langle f \rangle^+_{\Lambda_1;\beta,h} \geq \langle f \rangle^+_{\Lambda_2;\beta,h}. \tag{121}$$

The same statement holds for the $-$ b.c. and a nonincreasing function f (up to replacing $\geq$ by $\leq$).

Before giving the proof of the lemma, let us briefly discuss a spatial Markov property satisfied by $\mu^\eta_{\Lambda;\beta,h}$. In the exercises you will prove that for all $\Delta \subset \Lambda \Subset \mathbb{Z}^d$ and all $\eta \in \Omega$, $\omega' \in \Omega^\eta_\Lambda$, we have

$$\mu^\eta_{\Lambda;\beta,h}\left(\cdot \mid \sigma_i = \omega'_i,\ \forall i \in \Lambda \setminus \Delta \right) = \mu^{\omega'}_{\Lambda;\beta,h}(\cdot). \tag{122}$$

Let

$$\partial^{ex}\Delta = \{ i \notin \Delta : \exists j \in \Delta,\ i \sim j \}. \tag{123}$$

The probability on the r.h.s. of (122) really only depends on ω'_i for $i \in \partial^{ex}\Delta$. This implies

$$\mu^{\eta}_{\Lambda;\beta,h}\big(A \mid \sigma_i = \omega'_i,\ \forall i \in \Lambda\setminus\Delta\big)$$

$$= \mu^{\eta}_{\Lambda;\beta,h}\big(A \mid \sigma_i = \omega'_i,\ \forall i \in \partial^{\mathrm{ex}}\Delta\big) \tag{124}$$

for all events A depending only on the spins inside Δ. In this sense, (122) is indeed a spatial Markov property.

Proof of *Lemma 3.22*. From (122) we know that

$$\langle f\rangle^{+}_{\Lambda_1;\beta,h} = \langle f \mid \sigma_i = 1,\ \forall i \in \Lambda_2\setminus\Lambda_1\rangle^{+}_{\Lambda_2;\beta,h}. \tag{125}$$

The indicator function $\mathbb{1}_{\{\sigma_i=1,\ \forall i\in\Lambda_2\setminus\Lambda_1\}}$ is a nondecreasing function. The FKG inequality thus implies that

$$\langle f\rangle^{+}_{\Lambda_1;\beta,h} = \frac{\langle f\,\mathbb{1}_{\{\sigma_i=1,\ \forall i\in\Lambda_2\setminus\Lambda_1\}}\rangle^{+}_{\Lambda_2;\beta,h}}{\langle \mathbb{1}_{\{\sigma_i=1,\ \forall i\in\Lambda_2\setminus\Lambda_1\}}\rangle^{+}_{\Lambda_2;\beta,h}}$$

$$\overset{\text{FKG}}{\geq} \frac{\langle f\rangle^{+}_{\Lambda_2;\beta,h}\,\langle \mathbb{1}_{\{\dots\}}\rangle^{+}_{\Lambda_2;\beta,h}}{\langle \mathbb{1}_{\{\dots\}}\rangle_{\Lambda_2;\beta,h}} = \langle f\rangle^{+}_{\Lambda_2;\beta,h}. \quad \blacksquare \tag{126}$$

The next lemma shows that the Gibbs distributions with + and − b.c. play an extremal role, in the sense that they maximally favor + and − spins, respectively.

Lemma 3.23. Let f be a nondecreasing function. For $\beta \geq 0$ and $h \in \mathbb{R}$ we have

$$\langle f \rangle^{-}_{\Lambda;\beta,h} \leq \langle f \rangle^{\eta}_{\Lambda;\beta,h} \leq \langle f \rangle^{+}_{\Lambda;\beta,h}, \tag{127}$$

for any $\eta \in \Omega$ and any $\Lambda \Subset \mathbb{Z}^d$. Similarly, if f is a local function with $\operatorname{supp}(f) \subset \Lambda$, resp. $\operatorname{supp}(f) \subset V_N$, then

$$\langle f \rangle^{-}_{\Lambda;\beta,h} \leq \langle f \rangle^{\phi}_{\Lambda;\beta,h} \leq \langle f \rangle^{+}_{\Lambda;\beta,h},$$

$$\langle f \rangle^{-}_{\Lambda;\beta,h} \leq \langle f \rangle^{per}_{V_N;\beta,h} \leq \langle f \rangle^{+}_{\Lambda;\beta,h}. \tag{128}$$

Proof: We start with (127) and define

$$\mathcal{J}(\omega) = \exp\Big(\beta \sum_{\substack{i\in\Lambda,\, j\notin\Lambda \\ i\sim j}} \omega_i (1-\eta_j)\Big). \tag{129}$$

For $\omega \in \Omega_\Lambda^+$ and $\tilde{\omega} \in \Omega_\Lambda^h$ with $\omega|_\Lambda = \tilde{\omega}|_\Lambda$ we have

$$\exp\left(-\mathcal{H}_{\Lambda;\beta,h}(\omega)\right) = \exp\left(-\mathcal{H}_{\Lambda;\beta,h}(\tilde{\omega})\right) \mathcal{J}(\tilde{\omega}). \tag{130}$$

Since f is nondecreasing, we also have

$$\sum_{\omega \in \Omega_\Lambda^+} \exp\left(-\mathcal{H}_{\Lambda;\beta,h}(\omega)\right) f(\omega) \geq \sum_{\omega \in \Omega_\Lambda^h} \exp\left(-\mathcal{H}_{\Lambda;\beta,h}(\omega)\right) \mathcal{J}(\omega) f(\omega). \tag{131}$$

This implies

$$\langle f \rangle_{\Lambda;\beta,h}^+ = \frac{\sum_{\omega \in \Omega_\Lambda^+} \exp\left(-\mathcal{H}_{\Lambda;\beta,h}(\omega)\right) f(\omega)}{\sum_{\omega \in \Omega_\Lambda^+} \exp\left(-\mathcal{H}_{\Lambda;\beta,h}(\omega)\right)}$$

$$\overset{(130),\,(131)}{\geq} \frac{\sum_{\omega \in \Omega_\Lambda^h} \exp\left(-\mathcal{H}_{\Lambda;\beta,h}(\omega)\right) \mathcal{J}(\omega) f(\omega)}{\sum_{\omega \in \Omega_\Lambda^h} \exp\left(-\mathcal{H}_{\Lambda;\beta,h}(\omega)\right) \mathcal{J}(\omega)} = \frac{\langle \mathcal{J} f \rangle_{\Lambda;\beta,h}^h}{\langle \mathcal{J} \rangle_{\Lambda;\beta,h}^h}$$

FGK ineq.

$$\underset{\uparrow}{\geq} \langle f \rangle_{\Lambda;\beta,h}^h . \tag{132}$$

Use that $\mathcal{J}$ is a nondecreasing function.

This proves the claim in this case.

The proof for the free b.c. is identical. Here we need to use the function

$$\mathcal{J}(\omega) = \exp\Big(\beta \sum_{\substack{i\in\Lambda, j\notin\Lambda \\ i\sim j}} \omega_i \Big). \tag{133}$$

For a picture that illustrates this proof see Friedli, Velenik Fig. 3.7.

It remains to consider the case with periodic b.c..

In that case we argue as in the proof of Lemma 3.22. We consider V_N as a subset of $\mathbb{Z}^d$ and define

$$\Sigma_N = \{ i = (i_1, \dots, i_d) \in V_N : \exists\ 1 \le k \le d \text{ s.t. } i_k = 0 \}. \tag{134}$$

For $\omega \in V_{N-1}^+$ we have

$$\mu^{per}_{V_N;\beta,h}\left(\omega|_{V_N} \;\middle|\; \sigma_i = 1 \ \forall\, i \in \Sigma_N\right) = \mu^+_{V_{N-1};\beta,h}(\omega). \tag{135}$$

The result follows easily from (135). $\blacksquare$

We are now prepared to prove the existence and translation invariance of $\langle \cdot \rangle^{+}_{\beta,h}$ and $\langle \cdot \rangle^{-}_{\beta,h}$.

Proof of Theorem 3.17.: We consider the case with + b.c.. Let f be a local function. An application of Lemma 3.18 shows that its expectation value in finite volume can be written as

$$\langle f \rangle^{+}_{\Lambda_n;\beta,h} = \sum_{A \subset \mathrm{Supp}(f)} \tilde{f}_A \langle n_A \rangle^{+}_{\Lambda_n;\beta,h}. \tag{136}$$

recall: $n_A = \prod_{j \in A} n_j$; $n_j = \frac{1}{2}(1+\sigma_j)$

The functions n_A are non-decreasing. From Lemma 3.22 we therefore know that

$$\langle n_A \rangle^{+}_{\Lambda_n;\beta,h} \geq \langle n_A \rangle^{+}_{\Lambda_{n+1};\beta,h}, \quad \forall n \geq 1. \tag{137}$$

The sequence in (137) is monotone decreasing and bounded from below, and therefore has a limit. It follows that

$\langle f \rangle^+_{\Lambda_n;\beta,h}$ also has a limit, which we denote by

$$\langle f \rangle^+_{\beta,h} = \lim_{n\to\infty} \langle f \rangle^+_{\Lambda_n;\beta,h} . \tag{138}$$

It is easy to check that the limit is a linear, positive and normalized linear functional. In other words, $\langle \cdot \rangle^+_{\beta,h}$ is a Gibbs state.

Next, we check that it does not depend on the sequence $\Lambda_n \uparrow \mathbb{Z}^d$. Let $\Lambda_n^1 \uparrow \mathbb{Z}^d$ and $\Lambda_n^2 \uparrow \mathbb{Z}^d$, and denote by $\langle \cdot \rangle^{+,1}_{\beta,h}$ and $\langle \cdot \rangle^{+,2}_{\beta,h}$ the corresponding limits. Since both seq. converge to $\mathbb{Z}^d$ we can find $\{\Delta_n\}_{n\geq 1}$ s.t., for all $k \geq 1$

$$\Delta_{2k-1} \subset \{\Lambda_n^1 : n \geq 1\}, \quad \Delta_{2k} \subset \{\Lambda_n^2 : n \geq 1\}, \quad \Delta_k \subsetneq \Delta_{k+1} . \tag{139}$$

Of course $\Delta_n \uparrow \mathbb{Z}^d$. We thus know that $\lim_{k\to\infty} \langle f \rangle^+_{\Delta_k;\beta,h}$ exists for every local function f. This allows us to conclude that

$$\lim_{n\to\infty} \langle f \rangle^{+}_{\Lambda_n^1;\beta,h} = \lim_{k\to\infty} \langle f \rangle^{+}_{\Delta_k;\beta,h} = \lim_{n\to\infty} \langle f \rangle^{+}_{\Lambda_n^2;\beta,h} \tag{140}$$

holds. Hence $\langle \circ \rangle^{+}_{\beta,h}$ does not depend on the choice of $\{\Lambda_n\}_{n\geq 1}$.

It remains to prove the translation invariance of $\langle \circ \rangle^{+}_{\beta,h}$.

For all $j \in \mathbb{Z}^d$, $f \circ \Theta_j$ is a local function if f is one.

Let $\Theta_i A = A + i$, $A \in \mathbb{Z}^d$. If $\Lambda_n \uparrow \mathbb{Z}^d$ we also have

$\Theta_{-j} \Lambda_n \uparrow \mathbb{Z}^d$. Hence,

$$\langle f \rangle^{+}_{\Lambda_n;\beta,h} \longrightarrow \langle f \rangle^{+}_{\beta,h} \quad \text{and} \quad \langle f \circ \Theta_j \rangle^{+}_{\Theta_{-j}\Lambda_n;\beta,h} \longrightarrow \langle f \rangle^{+}_{\beta,h}. \tag{141}$$

Since

$$\langle f \circ \Theta_j \rangle^{+}_{\Theta_{-j}\Lambda_n;\beta,h} = \langle f \rangle^{+}_{\Lambda_n;\beta,h} \tag{142}$$

the claim follows. $\blacksquare$

Remark: With similar arguments one can also construct Gibbs states using the free boundary condition.

3.7. Phase diagram

In this section we discuss the phase diagram of the Ising model. One main question is whether the influence of the $\pm$ b.c. survives in the thermodynamic limit, leading to multiple Gibbs states. The answer to this question depends on d, β and h.

Contrary to what often happens in mathematics, the lack of uniqueness is not a defect of this approach, but is actually one of its main features: lack of uniqueness means that providing a complete microscopic description of the system (that is, the set of configurations and the Hamiltonian) as well as fixing all relevant thermodynamic parameters (β and h) is <u>not</u> sufficient to completely determine the macroscopic behavior of the system.

Definition 3.24. If at least two Gibbs states can be constructed for a pair (β, h), we say that there is a first order phase transition at (β, h).

Later in this section (see Thm. 3.34), we will relate this probabilistic definition of a first order phase transition to an analytic one associated with the pressure.

Theorem 3.25. 1.) In any $d \geq 1$, when $h \neq 0$, there is a unique Gibbs state for all values of $\beta \in [0, \infty)$.

2.) In $d = 1$, there is a unique Gibbs state at each $(\beta, h) \in [0,\infty) \times \mathbb{R}$.

3.) When $h = 0$ and $d \geq 2$, there exists $\beta_c(d) \in (0, \infty)$ s.t.:

- when $\beta < \beta_c$, the Gibbs state is unique,
- when $\beta > \beta_c$, a first order phase transition occurs at $(\beta, 0)$:

$$\langle \cdot \rangle^{+}_{\beta,0} \neq \langle \cdot \rangle^{-}_{\beta,0} . \tag{143}$$

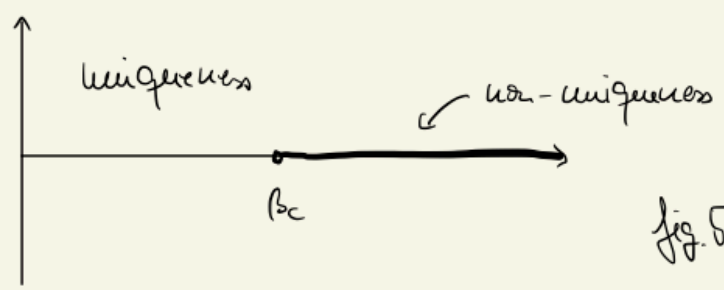


fig. 5

The proof of Thm. 3.25 is quite long and spread over several sections.

Remark 3.26. It can be proved that uniqueness also holds at $(\beta_c, 0)$ when $d \geq 2$. The argument is beyond the scope of this course.

Remark 3.27. The above Thm. claims the existence of two Gibbs states when $d \geq 2$ and $\beta > \beta_c$. It does, however, not describe the set of Gibbs states. This is a much more difficult problem.

3.7.1 Two criteria for (non)-uniqueness

In this section we provide a link between uniqueness of Gibbs states, the average magnetization density and differentiability of the pressure.

A first characterization of uniqueness

The major role played by $\langle \circ \rangle^+_{\beta,h}$ and $\langle \circ \rangle^-_{\beta,h}$ is made clear by the following statement.

Theorem 3.28. Let $(\beta,h) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$. The following statements are equivalent:

1. There is a unique Gibbs state at (β,h).
2. $\langle \cdot \rangle^+_{\beta,h} = \langle \circ \rangle^-_{\beta,h}$.
3. $\langle \sigma_0 \rangle^+_{\beta,h} = \langle \sigma_0 \rangle^-_{\beta,h}$.

Proof: The implications $(1) \Rightarrow (2) \Rightarrow (3)$ are trivial. Let us prove that $(3) \Rightarrow (2)$. We choose $\Lambda_n \uparrow \mathbb{Z}^d$ and $A \Subset \mathbb{Z}^d$.

We note that the function $\sum_{i\in A} n_i - u_A$ is non-decreasing. This will be proved in the exercises. An application of Lemma 3.23 therefore implies

$$\Big\langle \sum_{i\in A} n_i - u_A \Big\rangle^-_{\Lambda_n;\beta,h} \leq \Big\langle \sum_{i\in A} n_i - u_A \Big\rangle^+_{\Lambda_n;\beta,h}. \tag{144}$$

We rearrange the term, let $n\to\infty$, and find

$$\sum_{i\in A} \Big(\langle n_i\rangle^+_{\beta,h} - \langle n_i\rangle^-_{\beta,h} \Big) \geq \langle u_A\rangle^+_{\beta,h} - \langle u_A\rangle^-_{\beta,h}. \tag{145}$$

By translation invariance of $\langle\cdot\rangle^+_{\beta,h}$ and $\langle\cdot\rangle^-_{\beta,h}$, we have $\langle \sigma_i\rangle^\pm_{\beta,h} = \langle\sigma_0\rangle^\pm_{\beta,h}$. If (3) holds, this allows us to conclude that the l.h.s. of (145) vanishes. Hence,

$$\langle u_A\rangle^-_{\beta,h} \geq \langle u_A\rangle^+_{\beta,h}. \tag{146}$$

From Lemma 3.23 we know that also the reverse inequality holds. We conclude that

$$\langle u_A\rangle^+_{\beta,h} = \langle u_A\rangle^-_{\beta,h}. \tag{147}$$

Together with Lemma 3.15, this implies $\langle f\rangle^{+}_{\beta,h} = \langle f\rangle^{-}_{\beta,h}$ for every local function f. We have proven (2).

It remains to show that (2) $\Rightarrow$ (1). From Lemma 3.23 we know that any Gibbs state $\langle\cdot\rangle_{\beta,h}$ at (β,h) satisfies

$$\langle n_A\rangle^{-}_{\beta,h} \leq \langle n_A\rangle_{\beta,h} \leq \langle n_A\rangle^{+}_{\beta,h}. \tag{148}$$

If (2) holds, this implies equality in (148). With the same argument as above with see that this proves (1). ∎

Some properties of the magnetization density

We recall the definition of the magnetization density

$$m^{\#}_{\Lambda}(\beta,h) = \langle m_\Lambda\rangle^{\#}_{\Lambda;\beta,h} \quad \text{with} \quad m_\Lambda = \frac{1}{|\Lambda|}\sum_{i\in\Lambda} \sigma_i. \tag{149}$$

The uniqueness criterion in Thm. 3.28 is expressed in terms of $\langle\sigma_0\rangle^{\pm}_{\beta,h}$. It is natural to ask whether these quantities are related to $m^{\pm}_{\Lambda}(\beta,h)$.

Proposition 3.29. For any sequence $\Lambda \Uparrow \mathbb{Z}^d$, the limits

$$m^{\pm}(\beta,h) = \lim_{\Lambda_n \Uparrow \mathbb{Z}^d} m^{\pm}_{\Lambda}(\beta,h) \tag{150}$$

exist and

$$m^{\pm}(\beta,h) = \langle \sigma_0 \rangle^{\pm}_{\beta,h} . \tag{151}$$

Moreover, $h \mapsto m^+(\beta,h)$ is right-continuous and $h \mapsto m^-(\beta,h)$ is left-continuous.

Remark 3.30. From Corollary 3.7 we know that $m^+(\beta,h) = m(\beta,h)$ when $h \in \mathcal{B}_\beta$. Therefore,

$$m^*(\beta) = \lim_{\substack{h \downarrow 0 \\ h \in \mathcal{B}_\beta^c}} m(\beta,h) = \lim_{\substack{h \downarrow 0 \\ h \in \mathcal{B}_\beta^c}} m^+(\beta,h) = \langle \sigma_0 \rangle^+_{\beta,0} . \tag{152}$$

In the exercises you will show that

$$\lim_{\|i\|_1 \to \infty} \langle \sigma_0 \sigma_i \rangle^+_{\beta,0} = \left(\langle \sigma_0 \rangle^+_{\beta,0} \right)^2 = \left(m^*(\beta) \right)^2, \quad \forall \beta \geq 0. \tag{153}$$

This observation provides a convenient approach for the explicit computation of $m^*(\beta)$ in $d=2$. It avoids having to work with a non-zero magnetic field.

Proof of Proposition 3.29. Let $\Lambda_n \Uparrow \mathbb{Z}^d$. By the monotonicity property in Lemma 3.22. and the translation invariance of $\langle \circ \rangle^+_{\beta,h}$, we have

$$\langle \sigma_0 \rangle^+_{\beta,h} = \langle m_{\Lambda_n} \rangle^+_{\beta,h} \le \langle m_{\Lambda_n} \rangle^+_{\Lambda_n;\beta,h} . \tag{154}$$

Hence,

$$\liminf_{n\to\infty} \langle m_{\Lambda_n} \rangle^+_{\Lambda_n;\beta,h} \geqslant \langle \sigma_0 \rangle^+_{\beta,h} . \tag{155}$$

Next, we derive the reverse inequality. Let us fix $k \geqslant 1$ and $i \in \Lambda_n$ and denote by $B(k) = \{0, \dots, k-1\}^d$ the box with side length k. If $i + B(k) \subset \Lambda_n$ we know from Lemma 3.22 that

$$\langle \sigma_i \rangle^+_{\Lambda_n;\beta,h} \le \langle \sigma_i \rangle^+_{i+B(k);\beta,h} = \langle \sigma_0 \rangle^+_{B(k);\beta,h} . \tag{156}$$

On the other hand, if $i + B(k) \not\subset \Lambda_n$, then the box $i + B(k)$ intersects $\partial^{in}\Lambda_n$. As a consequence

$$\langle m_{\Lambda_n}\rangle^+_{\Lambda_n;\beta,h} = \frac{1}{|\Lambda_n|}\sum_{\substack{i\in\Lambda_n:\\ i+B(k)\subset\Lambda_n}}\langle \sigma_i\rangle^+_{\Lambda_n;\beta,h} + \frac{1}{|\Lambda_n|}\sum_{\substack{i\in\Lambda_n:\\ i+B(k)\not\subset\Lambda_n}}\langle \sigma_i\rangle^+_{\Lambda_n;\beta,h}$$

$$\leq \langle \sigma_0\rangle^+_{B(k);\beta,h} + \frac{|B(k)|\,|\partial^{in}\Lambda_n|}{|\Lambda_n|}. \tag{157}$$

$|\langle\sigma_i\rangle^+_{\Lambda_n;\beta,h}|\leq 1$

This implies

$$\limsup_{n\to\infty}\langle m_{\Lambda_n}\rangle^+_{\Lambda_n;\beta,h} \leq \langle\sigma_0\rangle^+_{B(k);\beta,h} \xrightarrow{k\to\infty} \langle\sigma_0\rangle^+_{\beta,h}, \tag{158}$$

and the desired result follows. The one-sided continuity of $m^\pm(\beta,h)$ will follow from Lemma 3.31 below. ∎

Lemma 3.31. 1. For all $\beta\geq 0$, $h\mapsto\langle\sigma_0\rangle^+_{\beta,h}$ is nondecreasing and right-continuous and $h\mapsto\langle\sigma_0\rangle^-_{\beta,h}$ is nonincreasing and left-continuous.

2. For all $h\geq 0$, $\beta\mapsto\langle\sigma_0\rangle^+_{\beta,h}$ is nondecreasing and, for all $h\leq 0$,

$\beta \mapsto \langle \sigma_0 \rangle^-_{\beta,h}$ is nonincreasing.

Proof: We prove the claims for $\langle \sigma_0 \rangle^+_{\beta,h}$. Symmetry then allows us to deduce those for $\langle \sigma_0 \rangle^-_{\beta,h}$.

1. Let $\Lambda \Subset \mathbb{Z}^d$. The FKG inequality implies

$$\frac{\partial}{\partial h} \langle \sigma_0 \rangle^+_{\Lambda;\beta,h} = \sum_{i \in \Lambda} \left(\langle \sigma_0 \sigma_i \rangle^+_{\Lambda;\beta,h} - \langle \sigma_0 \rangle^+_{\Lambda;\beta,h} \langle \sigma_i \rangle^+_{\Lambda;\beta,h} \right) \geq 0. \quad (159)$$

So, at fixed Λ, $h \mapsto \langle \sigma_0 \rangle^+_{\Lambda;\beta,h}$ is nondecreasing. One easily checks that this monotonicity persists in the thermodynamic limit.

Next, we show the right-continuity of $h \mapsto \langle \sigma_0 \rangle^+_{\beta,h}$. Let $\{h_m\}_{m \geq 1}$ be a sequence s.t. $h_m \downarrow h$ and $\{\Lambda_n\}_{n \geq 1}$ be a sequence s.t. $\Lambda_n \uparrow \mathbb{Z}^d$. In combination, Lemma 3.22 and (159) show that the double sequence $\{\langle \sigma_0 \rangle^+_{\Lambda_n;\beta,h_m}\}_{n,m \geq 1}$ is nonincreasing and bounded. It therefore follows from Lemma B.4 in Friedli, Velenik that

$$\lim_{m\to\infty} \langle \sigma_0 \rangle^+_{\beta,h_m} = \lim_{m\to\infty} \lim_{n\to\infty} \langle \sigma_0 \rangle^+_{\Lambda_n;\beta,h_m}$$

$$= \lim_{n\to\infty} \lim_{m\to\infty} \langle \sigma_0 \rangle^+_{\Lambda_n;\beta,h_m} = \lim_{n\to\infty} \langle \sigma_0 \rangle^+_{\Lambda_n;\beta,h} \qquad (160)$$

$$= \langle \sigma_0 \rangle^+_{\beta,h} .$$

$h \mapsto \langle \sigma_0 \rangle^+_{\Lambda_n;\beta,h}$ is continuous

(finite sum of continuous functions)

2. We have

$$\frac{\partial}{\partial\beta} \langle \sigma_0 \rangle^+_{\Lambda;\beta,h} = \sum_{\{i,j\}\in\mathcal{E}^b_\Lambda} \left(\langle \sigma_0 \sigma_i \sigma_j \rangle^+_{\Lambda;\beta,h} - \langle \sigma_0 \rangle^+_{\Lambda;\beta,h} \langle \sigma_i \sigma_j \rangle^+_{\Lambda;\beta,h} \right) \geq 0. \qquad (161)$$

GKS inequality

As above, the monotonicity of $\beta \mapsto \langle \sigma_0 \rangle^+_{\Lambda;\beta,h}$ in finite volume persists in the thermodynamic limit.

$\blacksquare$

Defining the critical temperature

We have $\langle \sigma_0 \rangle^-_{\beta,h} = - \langle \sigma_0 \rangle^+_{\beta,h}$ by symmetry. Theorem 3.28 and Remark 3.30 therefore imply that, when $h=0$, uniqueness is equivalent to $m^*(\beta)=0$. Since Lemma 3.31 implies that $m^*(\beta)=\langle\sigma_0\rangle^+_{\beta,0}$ is monotone in β, we are led naturally to the following definition.

Definition 3.32. The **critical inverse temperature** is

$$\beta_c(d) = \inf\{\beta\geq 0 : m^*(\beta)>0\} = \sup\{\beta\geq 0 : m^*(\beta)=0\}. \quad (162)$$

That is, $\beta_c(d)$ is the unique value of β s.t. $m^*(\beta)=0$ if $\beta<\beta_c$ and $m^*(\beta)>0$ if $\beta>\beta_c$. Of course, one still has to determine whether $\beta_c(d)$ is nontrivial, that is, whether $0<\beta_c(d)<+\infty$.

Remark 3.33. Translation invariance implies $\langle\sigma_i\rangle^+_{\beta,h}=\langle\sigma_0\rangle^+_{\beta,h}$ $=m^*(\beta)$ for all $i\in\mathbb{Z}^d$. Using the FGK inequality, we see that this implies

$$\langle\sigma_0\sigma_i\rangle^+_{\beta,0}\geq\langle\sigma_0\rangle^+_{\beta,0}\langle\sigma_i\rangle^+_{\beta,0} = m^*(\beta)^2. \quad (163)$$

In particular,

$$\inf_{i\in\mathbb{Z}^d}\langle\sigma_0\sigma_i\rangle^+_{\beta,0}\geq m^*(\beta)^2>0 \quad \forall\,\beta>\beta_c. \quad (164)$$

Such a behavior is referred to as **long-range order**. The presence of long-range order does, however, not imply that the random variables σ_i display strong correlations at large distances. Indeed, as we will show in the exercises, for any β,

$$\lim_{\|i\|_1 \to \infty} \left[\langle \sigma_0 \sigma_i \rangle^+_{\beta,h} - \langle \sigma_0 \rangle^+_{\beta,h} \langle \sigma_i \rangle^+_{\beta,h} \right] = 0. \tag{165}$$

That is, σ_0 and σ_i are always asymptotically (as $\|i\|_1 \to \infty$) uncorrelated.

A second characterization of uniqueness

The following theorem provides us with a link between the two notions of first-order phase transition introduced in Definitions 3.8 and 3.24.

Theorem 3.34. The following identities hold for all values of $\beta \geq 0$ and $h \in \mathbb{R}$:

$$\frac{\partial \psi}{\partial h^+}(\beta,h) = m^+(\beta,h), \qquad \frac{\partial \psi}{\partial h^-}(\beta,h) = m^-(\beta,h). \tag{166}$$

In particular, $h \mapsto \Psi(\beta,h)$ is differentiable at h iff there exists a unique Gibbs state at (β,h).

Remark 3.35. One can also show that the pressure of the Ising model is always differentiable w.r.t. β.

Proof of Thm. 3.34. We recall that the set $\mathcal{B}_\beta$ of points of non-differentiability of the pressure is at most countable. Accordingly, for any $h \in \mathbb{R}$, there exists a sequence $h_k \downarrow h$ s.t. $h_k \notin \mathcal{B}_\beta$ for all $k \geq 1$. It then follows from Corollary 3.7. that

$$\frac{\partial \Psi}{\partial h^+}(\beta,h) = \lim_{h_k \downarrow h} m(\beta,h_k) = \lim_{h_k \downarrow h} m^+(\beta,h_k) = m^+(\beta,h). \qquad (167)$$

Note that we also used the right-continuity of $h \mapsto m^+(\beta,h)$.

By symmetry

$$\frac{\partial \Psi}{\partial h^-}(\beta,h) = - \frac{\partial \Psi}{\partial h^+}(\beta,-h) = -m^+(\beta,-h) = m^-(\beta,h). \qquad (168)$$

We conclude that

$$\frac{\partial \Psi}{\partial h}(\beta,h) \text{ exists} \iff m^+(\beta,h) = m^-(\beta,h). \qquad (169)$$

The conclusion follows because Prop. 3.29 and Thm. 3.28 imply

$$m^+(\beta,h) = m^-(\beta,h) \iff \langle \sigma_0 \rangle^+_{\beta,h} = \langle \sigma_0 \rangle^-_{\beta,h} \iff \text{uniqueness at } (\beta,0). \tag{170}$$

$\blacksquare$

In the following two sections, we prove item no. 3 of Thm. 3.25, which establishes, at $h=0$, distinct low- and high-temperature behaviors.

3.7.2 Continuous symmetry breaking at low temperatures

In this subsection, we will prove that $\beta_c(d) < +\infty$, for all $d \geq 2$. To that end, we will show that, uniformly in the size of Λ,

$$\mu^+_{\Lambda;\beta,0}(\sigma_0 = -1) \leq \delta(\beta), \tag{171}$$

where $\delta(\beta) \downarrow 0$ for $\beta \to \infty$. A consequence of (171) is the bound

$$\langle \sigma_0 \rangle^+_{\Lambda;\beta,0} = \mu^+_{\Lambda;\beta,0}(\sigma_0 = 1) - \mu^+_{\Lambda;\beta,0}(\sigma_0 = -1)$$

$$= 1 - 2\mu^+_{\Lambda;\beta,0}(\sigma_0 = -1)$$

$$\geq 1 - 2\delta(\beta). \tag{172}$$

When we fix β s.t. $1 - 2\delta(\beta) > 0$ and then take the thermodynamic limit $\Lambda \uparrow \mathbb{Z}^d$, we find

$$m^*(\beta) = \langle \sigma_0 \rangle^+_{\beta,0} > 0. \tag{173}$$

This proves $\beta_c(d) < \beta < +\infty$ (see Def. 3.32), that is, a first order phase transition occurs at low temperature.

Low-temperature representation

Consider the 2-dim. Ising model in $\Lambda \in \mathbb{Z}^d$ with $h = 0$ and + b.c.. At low temperatures we expect the Gibbs measure to be (in some sense) a perturbation of the ground state η^+ (all spins $= +1$). That is, in a typical configuration $\omega \in \Omega^+_\Lambda$ we expect only few spins to be flipped. It therefore makes sense to use lines separating collections of -1 spins as new variables, see fig. 6 below.

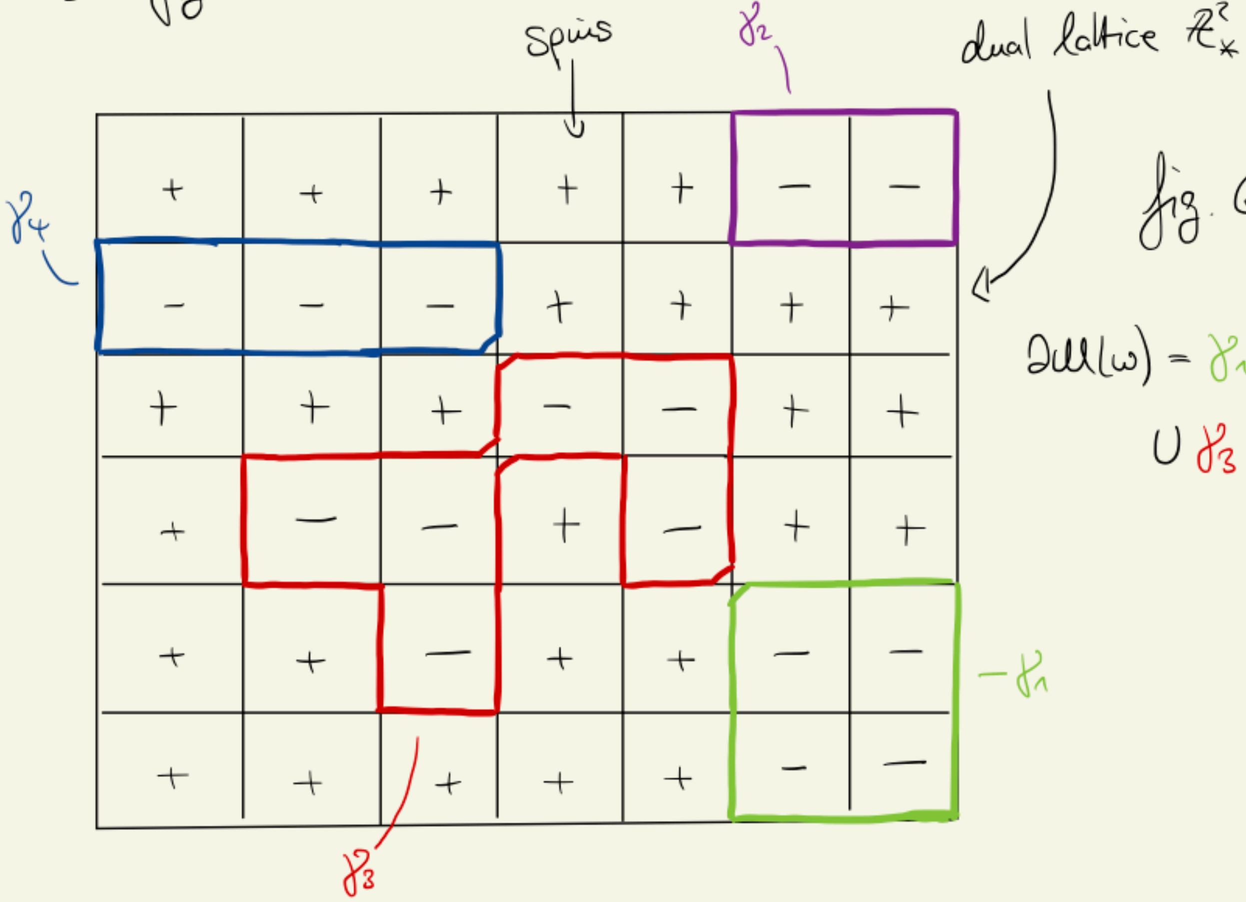


fig. 6

$\partial \mathcal{U}(\omega) = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$

We write the Hamiltonian in a way that emphasisis the role played by opposite spins:

$$\mathcal{H}_{\Lambda;\beta,0} = -\beta \sum_{\{i,j\}\in\mathcal{E}^b_\Lambda} \sigma_i\sigma_j = -\beta\,|\mathcal{E}^b_\Lambda| + \sum_{\{i,j\}\in\mathcal{E}^b_\Lambda} \beta(1-\sigma_i\sigma_j). \qquad (174)$$

The dependence on a state $\omega\in\Omega^+_\Lambda$ is only in the sum

$$\sum_{\{i,j\}\in\mathcal{E}^b_\Lambda} \beta\big(1-\sigma_i(\omega)\sigma_j(\omega)\big) = \sum_{\{i,j\}\in\mathcal{E}^b_\Lambda:\,\sigma_i(\omega)\neq\sigma_j(\omega)} 2\beta = 2\beta\,\#\big\{\{i,j\}\in\mathcal{E}^b_\Lambda : \sigma_i(\omega)\neq\sigma_j(\omega)\big\}. \qquad (175)$$

Let us associate with each vertex $i \in \mathbb{Z}^2$ the closed unit square centered at i:

$$S_i = i + \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]^2. \tag{176}$$

The boundary (in the sense of the standard topology on $\mathbb{R}^2$) of S_i, denoted by ∂S_i, can be considered as being made of four edges connecting nearest neighbors of the **dual lattice**

$$\mathbb{Z}^2_* = \mathbb{Z}^2 + \left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \left\{\left(i_1+\tfrac{1}{2}, i_2+\tfrac{1}{2}\right) : (i_1, i_2) \in \mathbb{Z}^2\right\}. \tag{177}$$

Notice that a given edge e of the original lattice intersects exactly one edge $e_\perp$ of the dual lattice. The curves in fig. 6 are made up by elements of the dual lattice. If we associate to each configuration $\omega \in \Omega^+_\Lambda$ the random set

$$\mathcal{M}(\omega) = \bigcup_{i \in \Lambda \,:\, \sigma_i(\omega) = -1} S_i \,, \tag{178}$$

then again $\partial\mathcal{M}(\omega)$ is made of edges of the dual lattice. Moreover, each edge $e_\perp = \{i,j\}_\perp \subset \partial\mathcal{M}(\omega)$ separates two opposite spins $\sigma_i(\omega) \neq \sigma_j(\omega)$. One can therefore write

$$\mathcal{H}_{\Lambda;\beta,0}(\omega) = -\beta\,|\mathcal{E}_\Lambda^b| + 2\beta\,|\partial\mathcal{M}(\omega)|. \tag{179}$$

Here $|\partial\mathcal{M}(\omega)|$ = length of $\partial\mathcal{M}(\omega)$ = number of edges contained in $\partial\mathcal{M}(\omega)$. A configuration ω with its associated set $\partial\mathcal{M}(\omega)$ is represented in fig. 6.

We will now decompose $\partial\mathcal{M}(\omega)$ into disjoint components. For that, it is convenient to fix a deformation rule to decide how these components are defined. We remark that each dual vertex of $\mathbb{Z}^2_*$ is adjacent to either 0, 2, or 4 edges of $\mathcal{M}(\omega)$. When this number is 4, we deform $\partial\mathcal{M}(\omega)$ using the rule

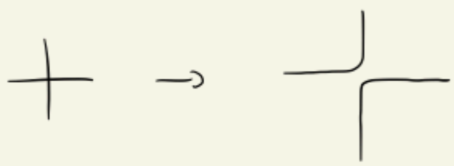


fig. 7.

An application of this rule at all such points yields a decomposition of $\partial\mathcal{M}(\omega)$ into a set of disjoint closed simple paths on the dual lattice, see fig. 6.

no repeating vertices

In terms of dual edges,

$$\partial\mathcal{M}(\omega) = \gamma_1 \cup \ldots \cup \gamma_n , \tag{180}$$

where each path γ_i is called a **contour of ω**. We define $\Gamma(\omega) = \{\gamma_1, \ldots, \gamma_n\}$ as well a the **length** $|\gamma|$ of a contour $\gamma \in \Gamma(\omega)$ as the number of edges of the dual lattice it contains.

Using this notation, we write

$$\mathcal{H}_{\Lambda;\beta,0}(\omega) = -\beta \, |\mathcal{E}_\Lambda^b| + 2\beta \sum_{\gamma \in \Gamma(\omega)} |\gamma| \tag{181}$$

for $\omega \in \Omega_\Lambda^+$. The partition function reads

$$Z^+_{\Lambda;\beta,0} = e^{\beta |\mathcal{E}_\Lambda^b|} \sum_{\omega \in \Omega_\Lambda^+} \prod_{\gamma \in \Gamma(\omega)} e^{-2\beta|\gamma|} , \tag{182}$$

and the probability of a configuration $\omega \in \Omega_\Lambda^+$ can be written as

$$\mu^+_{\Lambda;\beta,0}(\omega) = \frac{\exp\left(-\mathcal{H}_{\Lambda;\beta,0}(\omega)\right)}{Z^+_{\Lambda;\beta,0}} = \frac{\prod_{\gamma \in \Gamma(\omega)} e^{-2\beta|\gamma|}}{\sum_{\tilde{\omega} \in \Omega_\Lambda^+} \prod_{\gamma \in \Gamma(\tilde{\omega})} e^{-2\beta|\gamma|}} . \tag{183}$$

Remark 3.86. Note that the constant $e^{\beta|\mathcal{E}_\Lambda^b|}$ has cancelled in (183).

Peierls' argument

Let us consider the box $B(n) = \{-n, \dots, n\}^2$. Our goal is to obtain an upper bound for $\mu^+_{B(n);\beta,0}(\sigma_0 = -1)$. We first observe that any configuration $\omega \in \Omega^+_{B(n);\beta,0}$ with $\sigma_0(\omega) = -1$ must possess at least one (actually, an odd number of) contours surrounding the origin. Before we use this let us introduce the interior $\mathrm{Int}(\gamma)$ of a contour γ: each contour $\gamma \in \Gamma(\omega)$ is a bounded simple closed curve in $\mathbb{R}^2$ and therefore splits the plane into two regions. The one that is bounded is called the **interior** of γ and denoted by $\mathrm{Int}(\gamma)$. We have

$$\mu^+_{B(n);\beta,0}(\sigma_0 = -1) = \mu^+_{B(n);\beta,0}\left(\exists \gamma_* \in \Gamma : \mathrm{Int}(\gamma^*) \ni 0\right)$$

$$\leq \sum_{\gamma_* : \mathrm{Int}(\gamma_*) \ni 0} \mu^+_{B(n);\beta,0}(\gamma_* \in \Gamma) \tag{184}$$

Lemma 3.37. For any $\beta > 0$ and any contour γ_*,

$$\mu^+_{B(n);\beta,0}(\gamma_* \in \Gamma) \leq e^{-2\beta|\gamma_*|}. \tag{185}$$

Proof: We write

$$\mu^{+}_{B(n);\beta,0}(\gamma_* \in \Gamma) = \sum_{\omega:\, \gamma_* \in \Gamma(\omega)} \mu^{+}_{B(n);\beta,0}(\omega)$$

$$\overset{(183)}{=} e^{-2\beta|\gamma_*|} \frac{\displaystyle\sum_{\omega:\, \gamma_* \in \Gamma(\omega)} \prod_{\gamma \in \Gamma(\omega) \setminus \{\gamma_*\}} e^{-2\beta|\gamma|}}{\displaystyle\sum_{\omega \in \Omega^{+}_{B(n)}} \prod_{\gamma \in \Gamma(\omega)} e^{-2\beta|\gamma|}} . \qquad (186)$$

We will show that the numerator is bd. from above by the denominator by showing that the sum is the same, up to an additional constraint. To each configuration ω with $\gamma_* \in \Gamma(\omega)$ appearing in the sum in the numerator, we associate the configuration $\mathcal{E}_{\gamma_*}(\omega)$ obtained from ω by removing γ_*. This can be done by flipping all spins in the interior of γ_*:

$$\left(\mathcal{E}_{\gamma_*}(\omega)\right)_i = \begin{cases} -\omega_i & \text{if } i \in \mathrm{Int}(\gamma_*), \\ \omega_i & \text{otherwise.} \end{cases} \qquad (187)$$

The set of contours of $\mathcal{E}_{\gamma_*}(\omega)$ equals $\Gamma(\omega)\setminus\{\gamma_*\}$. Note that this is true even if $\mathrm{In}(\gamma_*)$ contains other contours. For an example, see fig. 8 below.

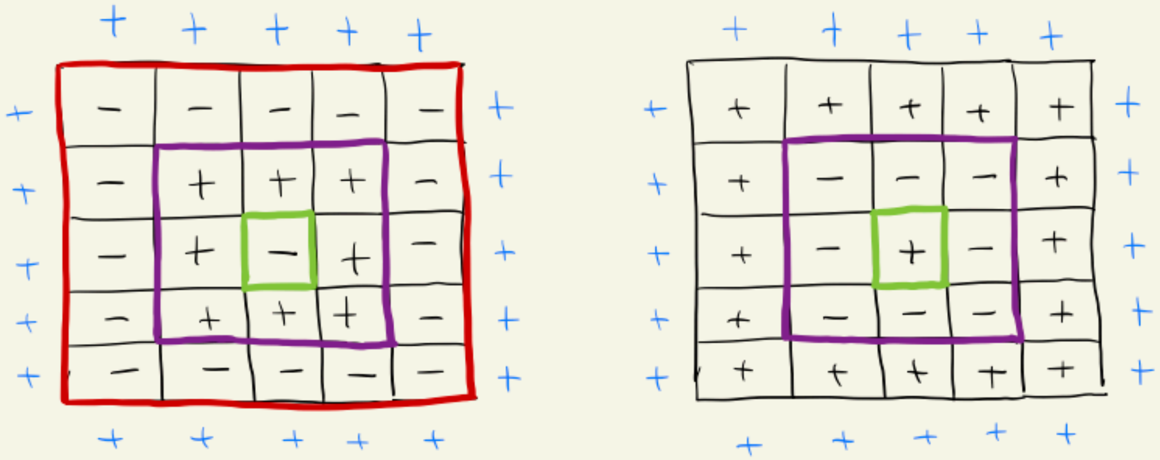


fig. 8

$\Gamma(\omega) = \{\gamma_1, \gamma_2, \gamma_*\}$ $\qquad$ $\Gamma\big(\mathcal{E}_{\gamma_*}(\omega)\big) = \{\gamma_1, \gamma_2\}$

We denote by $C(\gamma_*)$ the set of configurations that can be obtained by removing γ_* from a configuration containing γ_*. We have

$$\sum_{\omega:\, \gamma_* \in \Gamma(\omega)} \; \prod_{\gamma \in \Gamma(\omega)\setminus\{\gamma_*\}} e^{-2\beta|\gamma|} = \sum_{\omega' \in C(\gamma_*)} \; \prod_{\gamma' \in \Gamma(\omega')} e^{-2\beta|\gamma'|}. \tag{188}$$

Since the sum over $\omega' \in C(\gamma_*)$ contains fewer elements than the

Sum over all $\omega' \in \Omega^+_{B(n)}$. This shows that the ratio in (186) is bounded from above by 1.

$\blacksquare$

Remark: Each of the sums in (186) is typically exponentially large or small in $|B(n)|$. We have proved that their ratio is nevertheless bounded above by 1 by flipping the spins of the configuration inside the contour γ_*. Note that this operation relied crucially on the symmetry of the model under a global spin flip.

In the next step we will obtain a bound over all contours by grouping them according to their length and using the above lemma. We have

$$\mu^+_{B(n);\beta,0}(\sigma_0 = -1) \leq \sum_{\gamma_* : 0 \in \mathrm{int}(\gamma_*)} e^{-2\beta|\gamma_*|} =$$

$$= \sum_{k \geq 4} \sum_{\substack{\gamma_* : 0 \in \mathrm{Int}(\gamma_*), \\ |\gamma_*| = k}} e^{-2\beta |\gamma_*|}$$

minimal length of γ_* equals 4

$$= \sum_{k \geq 4} e^{-2\beta k} \, \# \{ \gamma_* : 0 \in \mathrm{Int}(\gamma_*), \ |\gamma_*| = k \}. \qquad (189)$$

A contour of length k surrounding the origin necessarily contains a vertex of the set $\{ (n - 1/2, 1/2) : n = 1, \dots, k/2 \}$. Moreover, the total number of contours starting from a given vertex is at most $4 \cdot 3^{k-1}$ (there are 4 options for the first segment and 3 for the remaining $k-1$). Hence,

$$\# \{ \gamma_* : 0 \in \mathrm{Int}(\gamma_*), \ |\gamma_*| = k \} \leq \frac{k}{2} \cdot 4 \cdot 3^{k-1}. \qquad (190)$$

Insertion of (190) into (189) yields

$$\mu^+_{B(n);\beta,0}(\sigma_0 = -1) \leq \frac{2}{3} \sum_{k \geq 4} \underbrace{k \, 3^k e^{-2\beta k}}_{} = \delta(\beta). \qquad (191)$$

$$= \exp\Big(k \big(\underbrace{-2\beta + \ln(3)}_{<0 \text{ for } 2\beta > \ln(3)} \big) + \ln(k) \Big)$$

The r.h.s. clearly converges to 0 for $\beta \to \infty$. This proves (171) and concludes the proof that $\beta_c(2) < +\infty$.

Extension to larger dimensions.

It remains to show that a phase transition also occurs for $d \geq 3$. To that end, one can adapt the Peierl's argument but the counting in (180) will be trickier. We investigate this route in the exercises. Here we will follow an alternative approach that uses an embedding of $\mathbb{Z}^d$ into $\mathbb{Z}^{d+1}$ and the GKS inequalities. The idea is very simple: one can built the Ising model on $\mathbb{Z}^{d+1}$ by considering a stack of Ising models on $\mathbb{Z}^d$ and adding interactions between neighboring spins in successive layers. The GKS inequality then tells us that adding these interactions does not decrease the magnetization. Accordingly, it does not increase the inverse critical temperature.

Let's be more precise. To avoid complicated notation, we focus on the case $d=3$. We introduce the notation

$$\mathcal{B}^3(n) = \{-n, \dots, n\}^3 \, , \qquad \mathcal{B}^2(n) = \{-n, \dots, n\}^2, \tag{192}$$

and claim that

$$\langle \sigma_0 \rangle^+_{\mathcal{B}^3(n);\beta,0} \geq \langle \sigma_0 \rangle^+_{\mathcal{B}^2(n);\beta,0} \tag{193}$$

holds. The proof goes as follows: consider the edges $\{i,j\} \subset \mathcal{B}^3(n)$ and assume that $i \sim j$, $i_3 = 0$, and $|j_3| = 1$. The two spins sitting at the endpoints of such an edge contribute to the total energy by an amount $-\beta \sigma_i \sigma_j = -J_{ij} \sigma_i \sigma_j$ (recall the notation in (95)). Thanks to the GKS inequality we have

$$\frac{\partial}{\partial J_{ij}} \langle \sigma_0 \rangle^+_{\mathcal{B}^3(n);\beta,0} = \langle \sigma_0 \sigma_i \sigma_j \rangle^+_{\mathcal{B}^3(n);\beta,0} - \langle \sigma_0 \rangle^+_{\mathcal{B}^3(n);\beta,0} \langle \sigma_i \sigma_j \rangle^+_{\mathcal{B}^3(n);\beta,0} \overset{\text{GKS}}{\geq} 0. \tag{194}$$

Hence, we can set this coupling equal to zero and we know that this can only decrease the expectation of σ_0.
This, of course, works for all other couplings between the layers. When we denote by $\langle \cdot \rangle^{+,0}_{\mathbb{B}^3(n);\beta,0}$ the Gibbs distributions with these couplings removed, this shows

$$\langle \sigma_0 \rangle^{+}_{\mathbb{B}^3(n);\beta,0} \geq \langle \sigma_0 \rangle^{+,0}_{\mathbb{B}^3(n);\beta,0} = \langle \sigma_0 \rangle^{+}_{\mathbb{B}^2(n);\beta,0}, \tag{185}$$

no interaction between layers

that is, (183). But this implies

$$\lim_{n\to\infty} \langle \sigma_0 \rangle^{+}_{\mathbb{B}^3(n);\beta,0} \geq \lim_{n\to\infty} \langle \sigma_0 \rangle^{+}_{\mathbb{B}^2(n);\beta,0}. \tag{186}$$

When combined with our previous results on the $d=2$ case, this proves the existence of a ferromagnetic phase for the model in $d=3$. An obvious adjustment of the above technique proves the claim for all $d \geq 2$. In particular,

$$\beta_c(d) \leq \beta_c(2) < +\infty. \tag{187}$$

Remark: It is known that

$$\beta_c(2) = \frac{1}{2} \operatorname{arcsinh}(1) \approx 0.441. \tag{198}$$

This follows from the exact solution of the model for $h=0$.

3.7.3 Uniqueness at high temperature

In this section we introduce a graphical representation of the Ising model that is well-suited to describe the correlations at small values of β.

High-temperature representation: Since spins take values in $\{\pm 1\}$ we have

$$\begin{aligned} e^{\beta \sigma_i \sigma_j} &= \cosh(\beta) + \sigma_i \sigma_j \sinh(\beta) \\ &= \cosh(\beta) \Big(1 + \tanh(\beta)\, \sigma_i \sigma_j \Big). \end{aligned} \tag{199}$$

For the exponential of the Hamiltonian, this implies

$$\exp\left(-\mathcal{H}_{\Lambda;\beta,0}(\omega)\right) = \prod_{\{i,j\}\in\mathcal{E}^b_\Lambda} e^{\beta\sigma_i\sigma_j} = \cosh(\beta)^{|\mathcal{E}^b_\Lambda|} \prod_{\{i,j\}\in\mathcal{E}^b_\Lambda} \left(1+\tanh(\beta)\,\omega_i\omega_j\right). \tag{200}$$

The following statement will be part of the exercises:

Exercise. Let $\mathcal{E}$ be a non-empty set. We then have and let $f:\mathcal{E}\to\mathbb{R}$ be some function. We then have

$$\prod_{e\in\mathcal{E}}\left(1+f(e)\right) = \sum_{E\subset\mathcal{E}} \prod_{e\in E} f(e). \tag{201}$$

Eqs. (200), (201) allow us to write the partition function as

$$Z^+_{\Lambda;\beta,0} = \cosh(\beta)^{|\mathcal{E}^b_\Lambda|} \sum_{E\subset\mathcal{E}^b_\Lambda} \tanh(\beta)^{|E|} \sum_{\omega\in\Omega^+_\Lambda} \underbrace{\prod_{\{i,j\}\in E} \omega_i\omega_j}_{=\prod_{i\in\Lambda}\omega_i^{J(i,E)}}, \tag{202}$$

where

$$J(i,E) = \#\left\{ j\in\mathbb{Z}^d : \{i,j\}\in E \right\} \tag{203}$$

denotes the incidence number. For the summation over ω, we have

$$\sum_{\omega_i = \pm 1} \omega_i^{J(i,E)} = \begin{cases} 2 & \text{if } J(i,E) \text{ is even} \\ 0 & \text{otherwise.} \end{cases} \tag{204}$$

Hence,

$$Z^+_{\Lambda;\beta,h} = 2^{|\Lambda|} \cosh(\beta)^{|\mathcal{E}^b_\Lambda|} \sum_{E \in K^{+,\text{even}}_\Lambda} \tanh(\beta)^{|E|}, \tag{205}$$

where

$$K^{+,\text{even}}_\Lambda = \{ E \subset \mathcal{E}^b_\Lambda : J(i,E) \text{ is even for all } i \in \Lambda \}. \tag{206}$$

When convenient, we will identify such sets of edges with the graph they induce. Eq. (205) is called the **high-temperature representation** of the partition function.

In the same way we see that $\langle \sigma_0 \rangle^+_{\Lambda;\beta,0}$ can be written as

$$\langle \sigma_0 \rangle^+_{\Lambda;\beta,0} = \left(Z^+_{\Lambda;\beta,0} \right)^{-1} 2^{|\Lambda|} \cosh(\beta) \sum_{E \in K^{+,0}_\Lambda} \tanh(\beta)^{|E|}$$

$$= \frac{\sum_{E \in K_\Lambda^{+,0}} \tanh(\beta)^{|E|}}{\sum_{E \in K_\Lambda^{+,\text{even}}} \tanh(\beta)^{|E|}}, \tag{207}$$

where

$$K_\Lambda^{+,0} = \left\{ E \subset \mathcal{E}_\Lambda^b \, : \, \mathcal{I}(i,E) \text{ for all } i \in \Lambda \setminus \{0\}, \text{ but } \mathcal{I}(0,E) \text{ is odd} \right\}. \tag{208}$$

Given $E \subset \mathcal{E}_\Lambda^b$, we denote by $\Delta(E)$ the set of all edges of $\mathcal{E}_\Lambda^b$ sharing no endpoint with an edge of E. Any collection of edges $E \in K_\Lambda^{+,0}$ can then be decomposed as $E = E_0 \cup E'$, with $E_0 \neq \emptyset$ the connected component of E containing 0, and $E' \in K_\Lambda^{+,\text{even}}$ satisfying $E' \subset \Delta(E_0)$. Therefore

$$\langle \sigma_0 \rangle^+_{\Lambda;\beta,0} = \sum_{\substack{E_0 \in K_T^{+,0} \\ \text{connected},\, 0 \in E_0}} \tanh(\beta)^{|E_0|} \frac{\sum_{E' \in K_\Lambda^{+,\text{even}},\, E' \subset \Delta(E_0)} \tanh(\beta)^{|E'|}}{\sum_{E \in K_\Lambda^{+,\text{even}}} \tanh(\beta)^{|E|}}. \tag{209}$$

Proof that $\beta_c(d) > 0$ for $d \geq 1$.

Estimating the second term on the r.h.s. of (208) we find

$$\langle \sigma_0 \rangle^{+}_{\Lambda,\beta,0} \leq \sum_{\substack{E_0 \in K_T^{+,0} \\ \text{connected},\ 0 \in E_0}} \tanh(\beta)^{|E_0|}. \tag{210}$$

To bound the sum on the r.h.s. we use the following lemma.

Lemma 3.38. Let G be a connected graph with N edges. Starting from an arbitrary vertex of G, there exists a path in G crossing each edge of G exactly twice.

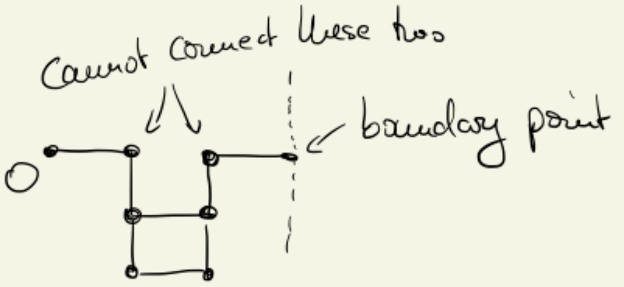
Proof: We will prove this statement by induction on N.

- $N = 1$. Here the result is trivial.
- Suppose the result holds when $N = k$ and let

  $$\pi = \big(\pi(1), \pi(2), \pi(3), \dots, \pi(2k)\big)$$

  be a path with the desired property. Now we add one more edge to the graph in such a way that it stays connected. This means that at least one endpoint v of this edge belongs to the original graph. We obtain the new path by following π until the first visit of v, then crossing the new edge in both directions, and finally following the path π to its end. $\blacksquare$

We come back to (210) and use the lemma to see that the number of graphs E_0 with l edges contributing to the sum is bounded from above by the number of paths of length $2l$ starting from 0. The latter is certainly smaller than $(2d)^{2l}$ since each new edge can be taken in at most $2d$ different directions. Moreover, E_0 necessarily connects 0 to $B(n)^c$. This follows from the fact that $\sum_{i \in \mathbb{Z}^d} J(i, E_0) = 2|E_0|$



is even; since $J(0, E_0)$ is odd there must be at least one vertex $i \neq 0$ with $J(i, E_0)$ odd; however, such a vertex cannot belong to $\mathcal{B}(n)$ since $\mathcal{J}(i, E_0)$ is even for $i \in \mathcal{B}(n) \setminus \{0\}$. This allows us to conclude that $|E_0| \geq n$. In combination with $\tanh(\beta) \leq \beta$, this implies

$$\langle G_0 \rangle^+_{\mathcal{B}(n); \beta, 0} \leq \sum_{\ell \geq n} \left(4d^2\beta\right)^\ell. \tag{211}$$

Let us choose β so small that $a = 4d^2\beta < 1$. Then we the r.h.s. of (211) equals

$$\sum_{\ell \geq n} a^\ell = a^n \sum_{\ell=0}^{\infty} a^\ell = \frac{a^n}{1-a}, \tag{212}$$

which implies

$$\lim_{n \to \infty} \langle G_0 \rangle^+_{\mathcal{B}(n); \beta, 0} = 0. \tag{213}$$

In particular,

$$\beta_c(d) > 0, \tag{214}$$

by Definition 3.32. From Theorem 3.28 we additionally know that there is only one unique Gibbs state at $(\beta, 0)$ with β as above.

3.7.4 Uniqueness in non-zero magnetic field

To finish the proof of Theorem 3.25, it still remains to prove item 1 (Note that the statements for $d=1$ follow from the exact solution in Section 3.3.). That is, we need to show that if $h \neq 0$ then the Gibbs state at (β, h) is always unique. To establish this result we will use tools from complex analysis.

Let us introduce the following domains in the complex plane:

$$\mathbb{H}^+ = \{ z \in \mathbb{C} \mid \mathrm{Re}(z) > 0 \},$$
$$\mathbb{H}^- = \{ z \in \mathbb{C} \mid \mathrm{Re}(z) < 0 \}. \tag{215}$$

Since β plays almost no role in this section we write $\psi(h)$ for the pressure.

Theorem 3.40. Let $\beta > 0$. The pressure $\psi(h)$ can be extended from $\{ h \in \mathbb{R} \mid h > 0 \}$ $(\{ h \in \mathbb{R} \mid h < 0 \})$ to an analytic function on $\mathbb{H}^+$ $(\mathbb{H}^-)$. On $\mathbb{H}^{+/-}$, ψ can be computed using the thermodynamic limit with free b.c..

Remark 3.41. This result, in particular, implies that $\frac{\partial \Psi}{\partial h}(h)$ exists for $h \in \mathbb{R} \setminus \{0\}$. By Theorem 3.34, this implies uniqueness of the Gibbs state for all $h \neq 0$, thus completing the proof of Theorem 3.25.

The partition function $Z^{\phi}_{\Lambda;\beta,h}$ is a linear combination of powers of $e^{\pm h}$, and therefore real analytic. Since $Z^{\phi}_{\Lambda;\beta,h} > 0$ for all $h \in \mathbb{R}$, the same is true for $\Psi^{\phi}_{\Lambda}(h) = \frac{1}{|\Lambda|} \ln Z^{\phi}_{\Lambda;\beta,h}$. It is not true, however, that this always holds after the thermodynamic limit has been taken: using Peierls' argument we have shown that the pressure is not differentiable at $h=0$ if β is chosen sufficiently small.

Since $Z^{\phi}_{\Lambda;\beta,h}$ is a complex number for $h \in \mathbb{C}$ it may also happen that it vanishes for some $h \in \mathbb{R}$, leading to the problem that the pressure would not even be well defined. Fortunately, This does not happen, as we will see below.

Theorem 3.42 (Lee Yang): Let $\beta \geq 0$. Let $D \subset \mathbb{C}$ be open, simply connected and s.t. $D \cap \mathbb{R}$ is an interval in $\mathbb{R}$. Assume that, for every finite set $\Lambda \Subset \mathbb{Z}^d$, we have

$$Z^{\phi}_{\Lambda;\beta h} \neq 0 \quad \forall\, h \in D. \tag{216}$$

Then, the pressure $h \mapsto \psi(h)$ admits an analytic continuation to D.

Proof: Let $\Lambda \Uparrow \mathbb{Z}^d$. In the following, we refer to Theorems of complex analysis, which can be found in the appendix of the book by Friedli and Velenik. From (216) and Theorem B.23 (existence of a function g s.t. $f = e^g$ for given f) that there exists an analytic function $h \mapsto \ln Z^{\phi}_{\Lambda_n;\beta h}$ on D that coincide with $|\Lambda|$ times the finite volume pressure when $h \in D \cap \mathbb{R}$ (see also remark below Thm). This allows us to define

$$g_n(h) = \exp\left(|\Lambda_n|^{-1} \ln Z^{\phi}_{\Lambda_n;(\beta,h)} \right), \tag{217}$$

which is analytic in D. It coincides with $\exp(\Psi^{\phi}_{\Lambda_n}(h))$ when $h \in D \cap \mathbb{R}$ and Thm. B.6. implies in this case $g_n(h) \to g(h) = \exp(\Psi(h))$ as $n \to \infty$. The sequence $\{g_n\}_{n=1}^{\infty}$ is locally uniformly bounded on D because

$$\left| Z^{\phi}_{\Lambda_n;\beta,h} \right| \leq \sum_{\omega \in \Omega_\Lambda} \exp\left(-\mathcal{H}^{\phi}_{\Lambda_n;\beta,\mathrm{Re}(h)}(\omega) \right)$$

$$\leq \exp\Big(\big(2d\beta + |\mathrm{Re}(h)| + \ln(2) \big) |\Lambda_n| \Big). \tag{218}$$

Hence, $|g_n(h)| = \exp\left(|\Lambda_n|^{-1} \ln | Z^{\phi}_{\Lambda_n;\beta,h} | \right) \leq \exp\left(2d\beta + |\mathrm{Re}(h)| + \ln(2) \right)$. An application of Thm. B.25 (Vitali's convergence theorem) shows that $\{g_n\}_{n=1}^{\infty}$ converges locally uniformly on D to an analytic function g. Moreover, since $g_n(h) \neq 0$ for all $h \in D$ and all $n \geq 1$, Theorem B.26 (Hurwitz theorem) implies that g has no zeros in D. From the fact that g has no zeros in D we conclude that there it admits an analytic logarithm in D. Choosing the branch that is real on $D \cap \mathbb{R}$, the function $\ln(g)$ coincides with Ψ on $h \in D \cap \mathbb{R}$. This proves the claim. $\blacksquare$

Before we can prove Thm. 3.42 using 3.40, we still need to show:

Theorem 3.43. (Lee-Yang circle Theorem): Condition (216) is satisfied when $D = \mathbb{H}^+$ and when $D = \mathbb{H}^-$.

Let us define $z = e^{-2h}$. We have $h \in \mathbb{H}^+$ iff $z \in \mathcal{U}$ with

$$\mathcal{U} = \{ z \in \mathbb{C} \mid |z| < 1 \}. \tag{219}$$

The above Thm. implies that all zeros of $Z^{\phi}_{\Lambda;\beta,h}$ lie in $\partial\mathcal{U}$ when viewed as a function of z (this explains the name of the theorem).

Proof: The claim is obvious for $\beta = 0$, so we focus on $\beta > 0$. In the following it will be convenient to think of the model as being defined on a subgraph (V,E) of $\mathbb{Z}^d$ with no isolated vertices. That is, E is the finite set of edges between interacting spins and V is the set of all endpoints of edges in E.

We write the partition function as

$$Z^{\phi}_{V;\beta,h} = \sum_{\omega\in\Omega_V} \prod_{\{i,j\}\in E} e^{\beta\sigma_i(\omega)\sigma_j(\omega)} \prod_{i\in V} e^{h\sigma_i(\omega)} \tag{220}$$

$$= e^{\beta|E|+h|V|} \sum_{\omega\in\Omega_V} \prod_{\{i,j\}\in E} e^{\beta(\sigma_i(\omega)\sigma_j(\omega)-1)} \prod_{i\in V} e^{h(\sigma_i(\omega)-1)}.$$

Let us also identify a configuration ω with the set

$$X(\omega) = \{ i\in V \mid \sigma_i(\omega) = -1 \}. \tag{221}$$

This allows us to write

$$\sum_{\omega\in\Omega_V} \prod_{\{i,j\}\in E} e^{\beta(\sigma_i(\omega)\sigma_j(\omega)-1)} \prod_{i\in V} e^{h(\sigma_i(\omega)-1)}$$

$$= \sum_{X\subset V} a_E(X)\, z^{|X|} \overset{\text{def.}}{=} P_E(z),$$

where $a_E(\emptyset) = 1 = a_E(V)$, and

$$a_E(X) = \prod_{\substack{\{i,j\}\in E \\ i\in X,\ j\in V\setminus X}} e^{-2\beta}. \tag{222}$$

Note that $0 \leq a_E(X) \leq 1$. We want to prove that

$$0 \neq Z^{\phi}_{V;\beta,h} = e^{\beta|E| + h|V|}\, P_E(z). \tag{223}$$

Since $e^{h|V|} \neq 0$ for $h \in \mathbb{H}^+$ it suffices to show that $P_E(z)$ does not vanish in $\mathcal{U}$.

In the next step we replace the one variable but high degree polynomial $P_E(z)$ by a multiple-variable but degree-one (in each variable) polynomial: let $z_V = \{z_i\}_{i\in V} \in \mathbb{C}^V$ and consider

$$\hat{P}_E(z_V) = \sum_{X \subset V} a_E(X) \prod_{i\in X} z_i \tag{224}$$

and note that $\hat{P}_E(z_V) = P_E(z)$ if $z_V = \{z\}_{i\in V}$. We claim that

$$|z_i| < 1,\ \forall i \in V \Rightarrow \hat{P}_E(z_V) \neq 0. \tag{225}$$

To proof of this claim will be carried out by induction.

Assume first that E consists of the single edge $\{i,j\}$.

We have $a_{\{i,j\}}(\{i\}) = e^{-2\beta} = a_{\{i,j\}}(\{j\})$, and hence

$$\hat{P}_E\left(z_{\{i,j\}}\right) = z_i z_j + e^{-2\beta}(z_i + z_j) + 1 \stackrel{!}{=} 0$$

$$\Leftrightarrow \quad z_i\left(z_j + e^{-2\beta}\right) = -1 - e^{-2\beta} z_j$$

$$\Leftrightarrow \qquad z_i = \frac{-1 - e^{-2\beta} z_j}{z_j + e^{-2\beta}} . \tag{227}$$

Check that $z_j = -e^{-2\beta}$
does not yield a
solution because $e^{-4\beta} \neq 1$
$(\beta > 0)$.

In the exercises you will show that $0 \leq e^{-2\beta} < 1$ implies that the Möbius transformation $z \mapsto -\dfrac{e^{-2\beta} z + 1}{z + e^{-2\beta}}$ interchanges the interior and the exterior of $\mathbb{U}$. This implies that $|z_i| > 1$ if $|z_j| < 1$, so that $\hat{P}_E(z_i, z_j)$ never vanishes when both $|z_i|, |z_j| < 1$.

Next, we assume that (225) holds for (V,E) and let $b = \{i,j\}$

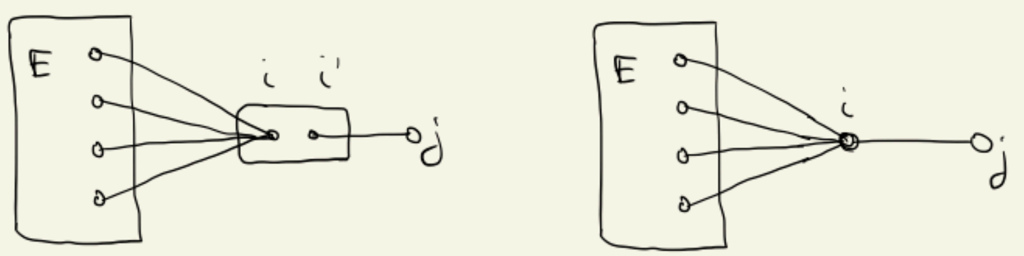
be an edge of $\mathbb{Z}^d$ not contained in E. We want to show that (225) holds for the graph $\left(V \cup \{i,j\}, E \cup \{b\}\right)$. In the following, we will consider three different cases.

Case 1: $V \cap \{i,j\} = \emptyset$. In this case we have

$$\hat{P}_{E\cup\{b\}}\left(z_{V\cup\{i,j\}}\right) = \hat{P}_E\left(z_V\right)\hat{P}_{\{b\}}\left(z_{\{i,j\}}\right). \tag{228}$$

By the induction hypothesis neither of the two polynomials on the r.h.s. vanishes when $|z_k| < 1$ for all $k \in V \cup \{i,j\}$. Hence, the same is true for their product.

Case 2: $V \cap \{i,j\} = \{i\}$. We will add the new edge in two steps: first, we add $b' = \{i', j\}$ to E with i' is a "virtual" vertex that is not present in V. Afterwards, we identify i' and i with a procedure called **Asano contraction**:



Since $V \cap \{i', j\} = \emptyset$ we are back to case 1: The polynomial $\hat{\mathcal{P}}_{E \cup \{b'\}}(z_{V \cup \{i', j\}})$ factorizes as in (228). Using the induction hypothesis, we conclude that it cannot vanish when all its variables have modulus smaller than 1.

Next, we split the sum over $X \subset V \cup \{i', j\}$ in $\hat{\mathcal{P}}_{E \cup \{b'\}}$ in several parts depending on whether $X \cap \{i', i\}$ equals $\{i, i'\}$, $\{i'\}$, $\{i\}$, or $\emptyset$:

$$\hat{\mathcal{P}}_{E \cup \{b'\}}(z_{V \cup \{i', j\}}) = \hat{\mathcal{P}}^{-,-} z_i z_{i'} + \hat{\mathcal{P}}^{+,-} z_{i'} + \hat{\mathcal{P}}^{-,+} z_i + \hat{\mathcal{P}}^{+,+} . \tag{229}$$

Here $\hat{\mathcal{P}}^{++}, \hat{\mathcal{P}}^{+,-}, \hat{\mathcal{P}}^{-,+}, \hat{\mathcal{P}}^{-,-}$ are polynomials in the remaining variables $z_j, z_k, k \in V \setminus \{i\}$.

The **Asano contraction** of $\hat{\mathcal{P}}_{E \cup \{b'\}}(z_{V \cup \{i', j\}})$ is defined as the polynomial

$$\hat{\mathcal{P}}^{-,-} z_i + \hat{\mathcal{P}}^{+,+} . \tag{230}$$

As we show in the next lemma, this polynomial coincides with $\hat{\Phi}_{E\cup\{b\}}(z_{V\cup\{j\}})$.

Lemma 3.44. Let $\tilde{V} = (V \setminus \{i\}) \cup \{j\}$. For $\sigma_1, \sigma_2 \in \{-,+\}$, we have

$$\hat{\Phi}^{\sigma_1,\sigma_2} = \sum_{X \subset \tilde{V}} a^{\sigma_1,\sigma_2}_{E\cup\{b\}}(X) \prod_{k\in X} z_k, \tag{231}$$

where

$$a^{-,-}_{E\cup\{b\}}(X) = \left(\mathbb{1}(j\in X) + \mathbb{1}(j\notin X)e^{-2\beta}\right) a_E(X\cup\{i\}),$$

$$a^{+,-}_{E\cup\{b\}}(X) = \qquad \text{"} \qquad a_E(X),$$

$$a^{-,+}_{E\cup\{b\}}(X) = \left(\mathbb{1}(j\notin X) + \mathbb{1}(j\in X)e^{-2\beta}\right) a_E(X\cup\{i\}),$$

$$a^{+,+}_{E\cup\{b\}}(X) = \qquad \text{"} \qquad a_E(X). \tag{232}$$

(Please check this!)

A brief computation also shows

$$\hat{P}_{E\cup\{b\}}\left(z_{V\cup\{j\}}\right) = \hat{P}^- z_i + \hat{P}^+ , \tag{233}$$

where

$$\hat{P}^\sigma = \sum_{X\subset \tilde{V}} a^\sigma_{E\cup\{b\}}(X) \prod_{k\in X} z_k , \quad \sigma\in\{-,+\}, \tag{234}$$

with

$$\begin{aligned} a^-_{E\cup\{b\}} &= \left(\mathbb{1}(j\in X) + \mathbb{1}(j\notin X)\, e^{-2\beta}\right) a_E\left(X\cup\{i\}\right), \\ a^+_{E\cup\{b\}} &= \left(\mathbb{1}(j\notin X) + \mathbb{1}(j\in X)\, e^{-2\beta}\right) a_E(X). \end{aligned} \tag{235}$$

In combination, (231) and (234) prove the claim. ∎

From the first paragraph on p. 126 we know that $\hat{P}_{E\cup\{b'\}}\left(z_{V\cup\{i',j\}}\right)$ cannot vanish when all its variables have modulus smaller than 1. We now use this property to infer that its Asano contraction also cannot vanish if the same holds for its variables.

Let us fix the variables z_k, $k \in V \setminus \{i\}$, and z_j so they all belong to $\mathcal{U}$. We know that $\hat{\Phi}_{E \cup \{b'\}}(z_{V \cup \{i', j\}})$ cannot vanish when z_i and $z_{i'}$ also belong to $\mathcal{U}$. Eq. (228) implies that

$$z \mapsto \hat{\Phi}^{-,-} z^2 + \left(\hat{\Phi}^{-,+} + \hat{\Phi}^{+,-} \right) z + \hat{\Phi}^{+,+} \tag{286}$$

cannot vanish if $|z| < 1$. Let $v, w \in \mathcal{U}$ be its roots. We have

$$\begin{aligned} \underline{\hat{\Phi}^{-,-} z^2} + \left(\underline{\hat{\Phi}^{-,+} + \hat{\Phi}^{+,-}} \right) z + \underline{\hat{\Phi}^{+,+}} &= \hat{\Phi}^{-,-} (z - v)(z - w) \\ &= \underline{\hat{\Phi}^{-,-} z^2} - \underline{\hat{\Phi}}^{-,-} z \underline{(v + w)} + \underline{\hat{\Phi}^{-,-} vw} \end{aligned}$$

$$\Rightarrow \quad \hat{\Phi}^{-,+} + \hat{\Phi}^{+,-} = \hat{\Phi}^{-,-} (v + w) \quad \text{and} \quad \hat{\Phi}^{+,+} = \hat{\Phi}^{-,-} vw. \tag{237}$$

With $|vw| > 1$ we infer $|\hat{\Phi}^{+,+}| > |\hat{\Phi}^{-,-}|$, which implies that

$$z \mapsto \hat{\Phi}^{-,-} z + \hat{\Phi}^{+,+} \tag{238}$$

cannot vanish when $|z|<1$.

<u>Case 3:</u> $V \cap \{ij\} = \{ij\}$. This case can be treated similarly. When we add a virtual edge $b'' = \{i', j'\}$ then the polynomial $\hat{P}_{E \cup \{b''\}}(z_{V \cup \{i', j'\}})$ satisfies (225) by Case 1. The claim follows when we apply two consecutive Asano contractions, the first to identify $z_{j'}$ and z_j, and the second to identify $z_{i'}$ and z_i.

$\blacksquare$

Remark 3.45. The above proof still works when we replace β by more general coupling $J_{ij} \geq 0$. Therefore, Theorem 3.42 can be adapted to obtain analyticity of the pressure in more general settings.

3.7.5. Summary

Let us summarize our findings in the following theorem.

Theorem 3.46. Let $\beta_c(d)$ be the inverse critical temp. of the Ising model on $\mathbb{Z}^d$.

1.) We have $\beta_c(1) = +\infty$, while $0 < \beta_c(d) < +\infty$ for $d \geq 2$.

2.) For all $\beta < \beta_c(d)$, the magnetization density $m(\beta,h)$ is well defined (and indep. of the b.c.s. and the sequence of sets in its definition) for all $h \in \mathbb{R}$. It is an odd, nondecreasing, continuous function of h. In particular, $m(\beta,0) = 0$.

3.) For all $\beta > \beta_c(d)$, $m(\beta,h)$ is well defined for all $h \in \mathbb{R}\setminus\{0\}$. It is an odd, non-decreasing function of h, which is continuous everywhere except at $h=0$, where

$$\lim_{h \downarrow 0} m(\beta,h) = m^+(\beta) > 0 \quad , \quad \lim_{h \uparrow 0} m(\beta,h) = m(\beta) < 0 . \qquad (259)$$

In particular, the spontaneous magnetization satisfies

$$m^*(\beta) = 0 \quad \text{if} \quad \beta < \beta_c(d), \qquad m^*(\beta) > 0 \quad \text{when} \quad \beta > \beta_c(d).$$

Remark 3.47. It is known that $m^*(\beta_c) = 0$. This can be used to show that $\beta \mapsto m^*(\beta)$ is continuous at β_c.

Remark 3.48. The above shows that, when $h = 0$, the spontaneous magnetization $m^*(\beta)$ allows to distinguish the ordered and the unordered phase of the model. A function with this property is called an **order parameter**.

Proof of Thm 3.46. The proof is elementary given our previous results, and therefore left to the reader.

Note that we have also seen that the phase transition can be described in terms of the Gibbs states.