6. Relation between BCS and Ginzburg-Landau theory

The Ginzburg-Landau functional reads

$$\mathcal{E}_D(\psi) = \frac{1}{|Q_a|}\int_{Q_a}\Big\{\Lambda_0 \left|(-i\nabla+2A(x))\psi(x)\right|^2 + \Lambda_1 W(x)\,|\psi(x)|^2 - D\Lambda_2\,|\psi(x)|^2 + \Lambda_3\,|\psi(x)|^4\Big\}\,dx \qquad (51)$$

with $\Lambda_0,\Lambda_2,\Lambda_3>0$; $D,\Lambda_1\in\mathbb{R}^3$, and $\psi\in H^1_{mag}(Q) = \{f\in L^2(Q)\mid T_{2B}(v)f = f \text{ for all } v\in a\mathbb{Z}^3,\ \|(-i\nabla+2A)\psi\|_2<+\infty\}$, and

$$E^{GL}(D) = \inf\{\mathcal{E}_D(\psi)\mid \psi\in H^1_{mag}(Q)\}. \qquad (52)$$

The GL functional has been introduced by Ginzburg and Landau in 1950 as a phenomenological theory of superconductivity, see

[GL] V. Ginzburg, L. Landau, On the theory of superconductivity, Zh. Eksp. Teor. Fiz. 20 (1950), 1064-1082.

A relation between the microscopic BCS theory and the macroscopic GL theory was established by Gor'kov in 1959 in

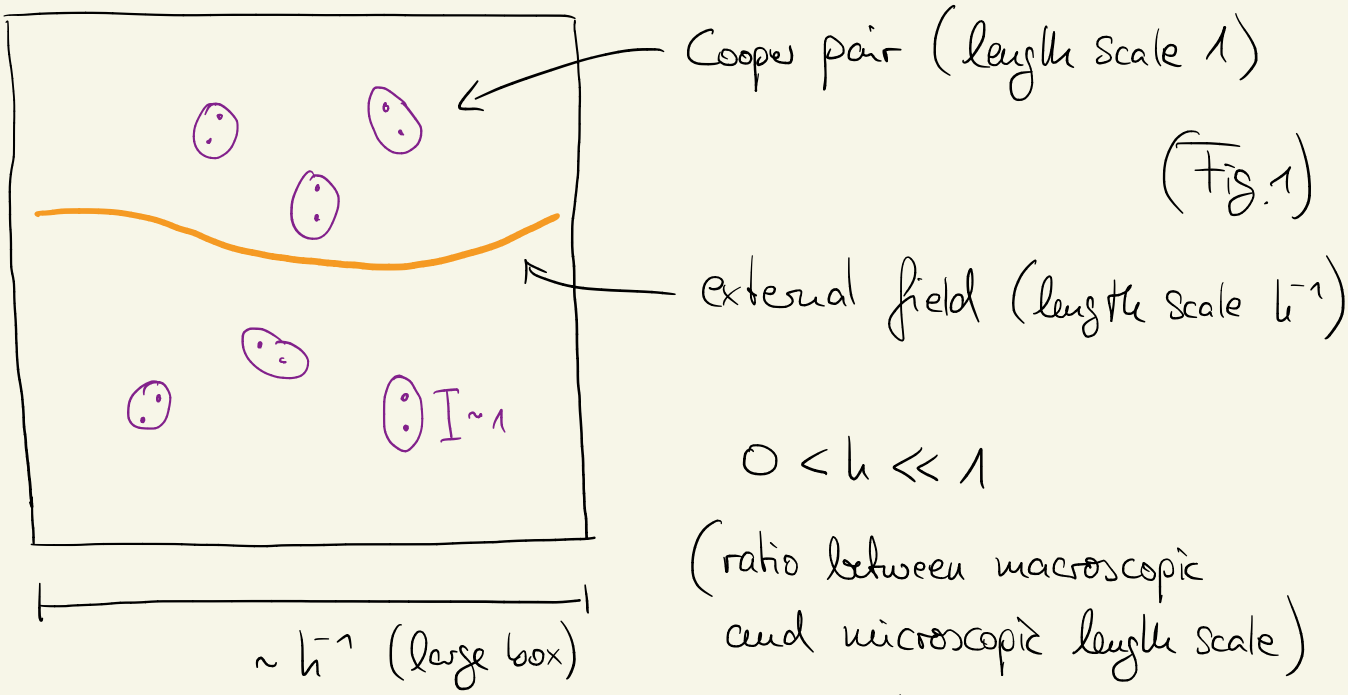
[Gor.] L.P. Gor'kov, Microscopic derivation of the Ginzburg-Landau equations in the theory of superconductivity, Zh. Eksp. Teor. Fiz. 36 (1959).

He showed that, close to the critical temperature, where the order parameters are expected to be small, GL theory arises when the free energy is expanded in powers of the gap function $\Delta(x,y) = 2V(x-y)\,\alpha(x,y)$.

The first mathematically rigorous proof of this relation was given in

[FHSS 2012] R.L. Frank, C. Hainzl, R. Seiringer, J.P. Solovej, Microscopic derivation of Ginzburg-Landau theory, J. Amer. Math. Soc. 25 (2012), 667-713.

The authors showed that in the presence of weak and macroscopic external fields, the macroscopic variations of the Cooper pair wave function of the system are correctly described by GL theory if the temperature is close to the critical temperature of the sample in an appropriate sense. The precise setup is as follows:



$0 < h \ll 1$

(ratio between macroscopic and microscopic length scale)

- External fields: $h^2 W(hx)$, $hA(hx)$, $a \sim h^{-1}$ (lattice constant). Notation $Q \to Q_h$
- Temperature: $T = T_c(1 - h^2 D)$ with $D \in \mathbb{R}$.

In non-technical terms, the main result in [FHSS 2012] is:

- $$\inf_{\Gamma} \mathcal{F}(\Gamma) - \underbrace{\mathcal{F}(\Gamma_0)}_{\text{free energy of normal state}}{}^{\sim 1 \text{ as } h\to 0} = h^4 \left(\inf_{\psi} \mathcal{E}_D^{GL} + o(1) \right) \qquad (53)$$

- The Cooper pair wave function α of any approximate minimizing state Γ of the BCS functional is of the form

$$\alpha(x,y) = h \underbrace{\alpha_*(x-y)}_{\text{related to translation invariant problem}} \underbrace{\psi\left(\frac{h(x+y)}{2}\right)}_{\text{approximate minimizer of GL functional}} + \text{l.o.} \qquad (54)$$

Later, the same mathematical framework has been used in

[FHSS 2016] R.L. Frank, C. Hainzl, R. Seiringer, J.P. Solovej, The external field dependence of the BCS critical temperature, Commun. Math. Phys. 342 (2016), 185–216.

to show that the BCS critical temperature shift caused by the external fields is of the form

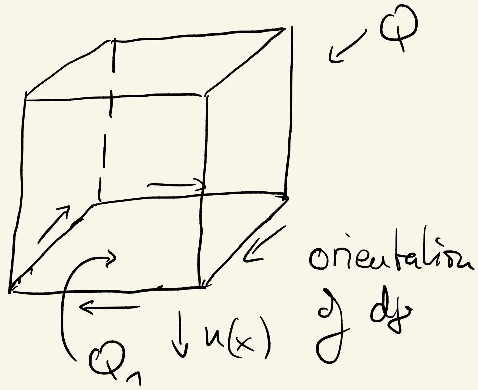
$$T_c(h) = \underset{\substack{\uparrow \\ \text{Critical temperature of the translation} \\ \text{invariant model}}}{T_c}\left(1 - D_c h^2\right) + o(h^2). \tag{55}$$

Here D_c denotes a critical parameter coming from linearized GL theory.

The main restriction in these works is that only periodic magnetic vector potentials are considered, which implies zero magnetic flux through the faces of the unit cell. This can be seen with an application of Stokes theorem:

$$\int_{Q_1} \underbrace{B(x)}_{\operatorname{rot} A(x)} \cdot n(x)\, dx \overset{\text{Stokes theorem}}{=} \int_{\partial Q_1} A(x) \cdot d\gamma(x) \underset{\substack{\uparrow \\ \text{use periodicity} \\ \text{of } A}}{=} 0.$$

line integral



(56)

An important step towards an extentions of the results in [FHSS 2016] to the case of magnetic fields with a non-zero flux through the unit cell has been provided in

[FHL 2019] R.L. Frank, C. Hainzl, E. Langmann, The BCS critical temperature in a weak homogeneous magnetic field, J. Spectr. Theory 9 (2019), 1005–1062.

There the problem of computing the BCS critical temperature shift in the presence of a weak homogeneous magnetic field has been considered within linearized BCS theory. The paper contains several important technical advances.

Recently, the results in [FHSS 2012, 2016] have been extended in

[DHM 2023a] A. D., C. Hainzl, M. O. Maier, Microscopic derivation of Ginzburg-Landau theory and the BCS critical temperature shift in a weak homogeneous magnetic field, Prob. Math. Phys. 4 (1) (2023), 103056

to the case of a constant magnetic field and in

[DHM 2023b] A. D., C. Hainzl, M. O. Maier, Microscopic derivation of Ginzburg-Landau theory and the BCS critical temperature shift for general external fields, Calc. Var. Partial Differ. Equ. 62, 203 (2023)

to the case of general external electric and magnetic fields (with nonzero flux through the unit cell).

The main novelty of these two works are a-priori estimates for low-energy states that include a constant magnetic field. The main difference between the systems with and without a constant magnetic field is that the components of the magnetic momentum operator $-i\nabla + A(x)$ do not commute while this is the case for the usual momentum operator $-i\nabla$. While in the latter case one can apply tools from

Fourier analysis, they are not available in the former case. We will discuss this issue in more detail later.

A precise mathematical statement relating the BCS and the GL functionals is provided in Theorems 2 and 3 below. Before we state them, we introduce the BCS free energy

$$F^{BCS}(h,D) = \inf\{ \mathcal{F}_h(\Gamma) - \mathcal{F}_h(\Gamma_0) \mid \Gamma \text{ admissible} \}.$$

BCS functional with $hA(hx)$, $h^2W(hx)$ as external potentials and with

$$T = T_c(1-h^2D),\ D\in\mathbb{R}. \tag{57}$$

The eigenfunction corresponding to the lowest eigenvalue of $K_{T_c}+U$ (in the following I omit the subscript μ) will be denoted by α_*, i.e.,

$$(K_{T_c}+U)\,\alpha_* = 0. \tag{58}$$

For Theorems 2 and 3 to hold, we need the following two assumptions.

Assumption 1: Let V be a radial function that satisfies $(1+|\cdot|^2)V \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. Moreover, let $W \in W^{1,\infty}(\mathbb{R}^3)$ and $A_{per} \in W^{3,\infty}(\mathbb{R}^3, \mathbb{R}^3)$ be periodic functions and assume that $A(0) = 0$.

Remark 6: If one wants to let the system choose the magnetic field self-consistently one needs to add the field energy

$$\frac{1}{|Q_h|} \int_{Q_h} |\operatorname{curl} A(x) - B_{ext}(x)|^2 \, dx, \tag{59}$$

where $B_{ext}(x)$ denotes an external magnetic field, to the BCS free energy functional. Now, one minimizes over the pair (Γ, A). It is clear that in this formulation regularity theory is needed in order to satisfy the assumptions (for three derivatives) for the magnetic vector potential to carry out our analysis for the matter part of the system.

Assumption 2: We assume that the interaction potential is such that the following holds:

(a) We have $T_c > 0$.

(b) The lowest eigenvalue of $K_{T_c} + V$ is simple.

A derivation of GL theory without Assumption 2.(b) and in the absence of external fields (translation invariant case) has been given in

[FraLem 2016] R.L. Frank, M. Lemm, Multi-component Ginzburg-Landau theory: microscopic derivation and examples, Ann. Henri Poincaré 17, 2285-2340 (2016).

The first theorem concerns an asymptotic expansions of the free energy of the free energy and the Cooper pair wave function.

Theorem 2: Let Assumptions 1 and 2 hold and let the coefficients $\Lambda_0, \Lambda_1, \Lambda_2$ and Λ_3 be given as in (72)-(75) below. Then we have

$$F^{BCS}(h, D) = h^4 \left(E^{GL}(D) + o(1) \right). \tag{60}$$

defined via $T = T_c(1 - h^2 D)$

Moreover, for any approximate minimizer Γ of $\mathcal{F}$ at $T = T_c(1 - Dh^2)$ in the sense that

$$\mathcal{F}(\Gamma) - \mathcal{F}(\Gamma_0) \leq h^4 \left(E^{GL}(D) + \rho \right) \tag{61}$$

holds for some $\rho > 0$, we have

$$\alpha(r, X) = \alpha_*(r)\,\psi(X) + \sigma(X, r) \tag{62}$$

for $\alpha = \Gamma_{12}$, and where σ satisfies

$$\frac{1}{|Q_h|} \int_{Q_h \times \mathbb{R}^3} |\sigma(r, X)|^2 \, d(X, r) \lesssim h^{11/3}. \tag{63}$$

box with lattice constant h^{-1}

The function Ψ obeys

$$\mathcal{E}(\Psi) \leq h^4 \left(E^{GL}(\Delta) + \rho + o(1) \right). \tag{64}$$

Remark 7: • It should be noted that $\mathcal{F}(\Gamma_0) \sim 1$.

This needs to be compared to the order h^4, on which the GL energy appears.

• We have

$$\frac{1}{|Q_h|} \int_{\mathbb{R}^3 \times Q_h} |\alpha_*(r) \Psi(X)|^2 \, d(r,X) \sim h^2. \tag{65}$$

The second theorem is a statement about the dependence of the BCS critical temperature on the external fields.

Theorem 3: Let Assumptions 1 and 2 hold. Then there are constants $C > 0$ and $h_0 > 0$ s.t. for all $0 < h < h_0$ the following holds:

(a) Let $0 < T_0 < T_c$. If

$$T_0 \leq T \leq T_c\left(1 - h^2\left(D_c + Ch^{1/2}\right)\right) \tag{66}$$

with

$$D_c = \frac{1}{\lambda_2} \inf \operatorname{spec}_{L^2_{mag}(Q)}\left(\lambda_0\left(-i\nabla + A\right)^2 + \lambda_1 W\right), \tag{67}$$

then we have

$$F^{BCS}(h, D) < 0. \tag{68}$$

(b) If

$$T \geq T_c\left(1 - h^2\left(D_c - Ch^{1/6}\right)\right) \tag{69}$$

then we have

$$\mathcal{F}(\Gamma) - \mathcal{F}(\Gamma_0) > 0, \tag{70}$$

unless $\Gamma = \Gamma_0$.

Remark 8: •) The interpretation of the above theorem is that the critical temperature of the full model satisfies

$$T_c(h) = T_c\left(1 - D_c h^2\right) + o(h^2). \tag{71}$$

The coefficient D_c is determined by linearized GL theory. Note however, that we do not know whether there exists a unique critical temperature. In principle, superconductivity could be present at a certain temperature, vanish if the temperature is increased, and reappear if it is further increased.

- We expect the assumption $0 < T_0 \leq T$, which also appears in [FHL 2015], to be only of technical nature.

The coefficients $\Lambda_0, \Lambda_1, \Lambda_2$ and Λ_3 read

- $$\Lambda_0 = \frac{1}{16 T_c^2} \int_{\mathbb{R}^3} \left\{ 2 |\widehat{V\alpha_*}(p)|^2 \left(g_1\left(\frac{p^2-\mu}{T_c}\right)\right) + \frac{2}{3 T_c} p^2 g_2\left(\frac{p^2-\mu}{T_c}\right) \right\} \frac{dp}{(2\pi)^3}, \tag{72}$$

- $$\Lambda_1 = \frac{1}{4 T_c^2} \int_{\mathbb{R}^3} 2 |\widehat{V\alpha_*}(p)|^2 g_1\left(\frac{p^2-\mu}{T_c}\right), \tag{73}$$

- $$\Lambda_2 = \frac{1}{8 T_c} \int_{\mathbb{R}^3} \frac{2 |\widehat{V\alpha_*}(p)|^2}{\cosh\left(\frac{(p^2-\mu)}{2T_c}\right)} \frac{dp}{(2\pi)^3}, \tag{74}$$

$$\dots \frac{1}{16 T_c^2} \int_{\mathbb{R}^3} |2 \widehat{V\alpha_*}(p)|^4 \frac{g_1\left(\frac{p^2-\mu}{T_c}\right)}{p^2-\mu} \frac{dp}{(2\pi)^3}, \qquad (75)$$

with

$$g_1(x) = \frac{\tanh(x/2)}{x^2} - \frac{1}{2x} \frac{1}{\cosh^2(x/2)} \qquad (76)$$

and

$$g_2(x) = \frac{1}{2x} \frac{\tanh(x/2)}{\cosh^2(x/2)}. \qquad (77)$$

In the remaining part of these notes we will discuss the proof of Theorem 2. We will first consider the case $A=0$ and then comment on the case $A \neq 0$. We start our discussion with the upper bound.

7. Upper bound for the BCS free energy

To prove the upper bound we need to find a trial state, whose free energy can be bounded from above by the r.h.s. of (60).

The Euler-Lagrange equation (also called Bogolubov-deGennes equation) of the BCS functional reads

$$\Gamma = \frac{1}{e^{H_\Delta/T}+1}, \qquad H_\Delta = \begin{pmatrix} k & \Delta \\ \bar{\Delta} & -k \end{pmatrix}, \tag{78}$$

with $k = (-i\nabla + A)^2 + W - \mu$. Here Δ is defined via its integral kernel by

$$\Delta(x,y) = -2V(x-y)\,\alpha(x,y), \qquad \alpha = [\Gamma]_{12}$$

(original coordinates)

$$\Delta(r,X) = -2V(r)\,\alpha(r,X). \tag{79}$$

(abuse of notation, relative and center of mass coordinates)

We want to show that α behaves to leading order as $\alpha_*(r)\psi(X)$, where ψ minimizes the GL functional, and hence we choose

$$\Gamma_\Delta = \frac{1}{e^{H_\Delta/T}+1} \quad \text{with} \quad \Delta(r,X) = -2V(r)\alpha_*(r)\psi(X) \tag{8o}$$

as trial state.

note that

$$\frac{1}{|Q_h|}\int_{Q_h} |\psi(x)|^2 dx \sim h^2$$

This is a small parameter!

To compute the free energy of Γ_Δ it is convenient to write

$$\mathcal{F}(\Gamma_\Delta) - \mathcal{F}(\Gamma_0) \tag{81}$$

$$= \frac{1}{2}\mathcal{H}_0(\Gamma_\Delta,\Gamma_0) + \frac{1}{|Q_h|}\int_{Q_h\times\mathbb{R}^3} V(r)\,|\alpha_\Delta(r,X)|^2\, d(r,X),$$

compare to (32)

where

$$\mathcal{H}_0(\Gamma_\Delta,\Gamma_0) = \frac{1}{2}\widetilde{\mathrm{tr}}_Q\left[\varphi(\Gamma_\Delta) - \varphi(\Gamma_0) - \varphi'(\Gamma_0)(\Gamma_\Delta-\Gamma_0)\right], \tag{82}$$

weak local trace

$$\widetilde{\mathrm{tr}}_Q[A] \overset{\downarrow}{=} \mathrm{tr}_Q\left[PAP + (1-P)A(1-P)\right] \quad \text{with} \quad P = \begin{pmatrix}1&0\\0&0\end{pmatrix}, \tag{83}$$

and $\varphi(x) = x\ln(x) + (1-x)\ln(1-x)$.

Using

$$\Gamma_\Delta = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{1}{2T} H_\Delta\right),$$

$$\ln(\Gamma_\Delta) = -\frac{1}{2T} H_\Delta - \ln\left(2\cosh\left(\frac{1}{2T} H_\Delta\right)\right) \tag{84}$$

we can rewrite (81) as (this is not entirely trivial, please check if you are interested or have a look at our paper)

$$\mathcal{F}(\Gamma_\Delta) - \mathcal{F}(\Gamma_0) = -\frac{1}{2T}\widetilde{\mathrm{Tr}}_{Q_h}\left[\ln\left(\cosh\left(\frac{1}{2T}H_\Delta\right)\right) - \ln\left(\cosh\left(\frac{1}{2T}H_0\right)\right)\right]$$

$$+ \|\psi\|^2_{L^2(Q_h)} \langle \alpha_*, V\alpha_* \rangle_{L^2(\mathbb{R}^3)}$$

$$+ \int_{Q_h \times \mathbb{R}^3} V(r) \left| \alpha(r,X) - \alpha_*(r)\psi(X) \right|^2 d(X,r). \tag{85}$$

We want to expand this first in powers of Δ and then in h.

Goal: Compute $\mathcal{F}(\Gamma_\Delta)$ with a resolvent expansion.

Tools: • Using $\cosh\left(\frac{1}{2T}x\right) = \prod_{k=0}^{\infty}\left(1 + \left(\frac{x}{\omega_k}\right)^2\right)$

with the Matsubara frequencies

$$\omega_n = \pi(2n+1)T, \quad n \in \mathbb{Z} \tag{86}$$

one can show that

$$\tilde{\int}_{Q_h} \left[\ln\left(\cosh\left(\frac{1}{2T} H_\Delta\right)\right) - \ln\left(\cosh\left(\frac{1}{2T} H_0\right)\right)\right] \tag{87}$$

$$= -i \sum_{k=0}^{\infty} \int_{\omega_k}^{\infty} \tilde{\int}_{Q_h} \left[\frac{1}{iu - H_\Delta} - \frac{1}{iu - H_0} + \frac{1}{iu + H_\Delta} - \frac{1}{iu + H_0} \right] du$$

holds.

The advantage of this approach in comparison to that in [FHSS 2012] is that all resolvents are evaluated at imaginary values and therefore contribute decay in n.

- Using

$$\tanh\left(\frac{1}{2T} H_\Delta\right) = -\frac{2}{T} \sum_{n\in\mathbb{Z}} \frac{1}{i\omega_n - H_\Delta} \tag{88}$$

and (84) we see that

$$\alpha_\Delta = [\Gamma_\Delta]_{12} = \frac{1}{T} \sum_{n\in\mathbb{Z}} \left[\frac{1}{i\omega_n - H_\Delta} \right]_{12}. \tag{89}$$

- **Resolvent identity**

$$\frac{1}{z - H_\Delta} = \frac{1}{z - H_0} + \frac{1}{z - H_0} (H_\Delta - H_0) \frac{1}{z - H_\Delta} \tag{90}$$

(Note that this identity can be iterated to set up an expansion.)

- We express all terms in the resolvent expansion in terms of their **integral kernels**. We e.g. have

$$\frac{1}{\underset{\uparrow\\ \omega\neq 0}{-\Delta-\mu+i\omega}}(x-y) = -\frac{1}{4\pi|x|}\exp\Big(-\underset{\substack{\nearrow\\ \text{standard branch of}\\ \text{square root}}}{\sqrt{-(i\omega+\mu)}}\,|x|\Big) \qquad (S1)$$

and the resolvent identity reads $(A=0)$

$$\frac{1}{-\Delta-\mu+W+i\omega}(x,y) = \frac{1}{-\Delta-\mu+i\omega}(x-y) \qquad (S2)$$

$$+\int_{\mathbb{R}^3}\frac{1}{-\Delta-\mu+i\omega}(x-z)\,W(z)\,\frac{1}{-\Delta-\mu+W+i\omega}(z,y)\,dz.$$

In case $A=0$ this formalism and the approach to expand in the operator formalism (note that different sets of norms are natural in the two formalism) are basically equivalent. However, If an external field is added to the problem, the approach via integral kernels is more useful. Let us briefly discuss two cases:

(a) Constant magnetic field B

Let $A(x) = \frac{1}{2} B e_z \wedge x$ and denote $h_B = (-i\nabla + A)^2$
($B \in \mathbb{R}$)
as well as

$$g_B^z(x) = \frac{1}{z - h_B}(x,0). \tag{93}$$

Then for all $B \geq 0$ and $z \in \mathbb{C} \setminus [B, \infty)$; $x, y \in \mathbb{R}^3$ we

$$\underbrace{\frac{1}{z - h_B}(x,y)}_{\text{not gauge invariant}} = e^{i \frac{Be_z}{2} \cdot (x \wedge y)} \, g_B^z(x-y). \tag{94}$$

Contains all the information we need to extract to obtain the GL energy (phase approximation). This factor is <u>not</u> gauge invariant.

Gauge invariant and very nicely behaved in perturbation theory for $0 < B \ll 1$. It is very helpful that this term depends only on $x-y$!

(b) General magnetic field

Let $A(x) = \frac{1}{2} B e_z \wedge x + A_{per}(x)$ and define

$$G_z(x,y) = \frac{1}{z - (-i\nabla + A) + \mu}(x,y) \; ; \quad x,y \in \mathbb{R}^3. \tag{95}$$

We also define the **non-integrable phase factor**, also called the **Wilson line**, by

$$\phi(x,y) = -\int_0^1 A(tx + (1-t)y)\cdot(x-y)\, dt. \tag{96}$$

It is convenient to write

gauge invariant

$$G_z(x,y) = e^{i\phi(x,y)}\, g_z(x,y). \tag{97}$$

Important: The function g_z and its gradient can both be bounded by functions that only depend on $x-y$. These functions then satisfy similar bounds than g_B^z in (84). This fact is crucial for our analysis.

The above facts and the way we do our analysis is an extension of the phase approximation method, which has been pioneered in the framework of linearized BCS theory and a constant magnetic field in [FHL 2019], to our nonlinear setting with periodic magnetic fields. The phase approximation method is a well-known tool in the physics literature, see e.g.

[HelWert 1966] E. Helfand, N.R. Werthamer, Temperature and Purity dependence of the Superconducting critical field, H_{c_2}. II. Phys. Rev. 147 (1966), 288–294,

and has also been used in the mathematical literature to study spectral properties of Schrödinger operators involving a magnetic field, for instance in

[CorNen 1998] H.D. Cornean, G. Nenciu, On eigenfunction decay for two-dimensional Schrödinger

operators, Commun. Math. Phys. 192 (1998), 671–685

and

[Nen 2002] G. Nenciu, On asymptotic perturbation theory for quantum mechanics: almost invariant subspaces and gauge invariant magnetic perturbation theory, J. Math. Phys. 43 (2002), 1273–1298.

Our approach in [DHM 2023a] and [DHM 2023b] should be compared to the trial state analysis in [FHSS 2012] and [FHSS 2016], where a semiclassical expansion is used. Also, the analysis in these two references uses a Cauchy integral representation of the function $z \mapsto \ln(1+\exp(-z))$, while our approach is based on a product expansion of the hyperbolic cosine in terms of Matsubara frequencies. In this way we obtain better decay properties in the subsequent resolvent expansion, which, in our opinion, simplifies the analysis considerably.

9. Lower bound for the BCS free energy

The proof of the lower bound for the BCS free energy is carried out in two main steps. Step 1 consists of proving the following theorem, which guarantees a decomposition of the Cooper pair wave function of any BCS state, whose free energy differs from that of the normal state only by a constant times h^4. It should be compared to that in Theorem 2. The precise statement reads

Theorem 4: Let Assumptions 1 and 2 hold. For given $D_0, D_1 \geq 0$, there exists $h_0 > 0$ s.t. for all $0 < h < h_0$ the following holds: if $T > 0$ obeys $T - T_c \geq -D_0 h^2$ and if Γ is an admissable BCS state with

$$\mathcal{F}(\Gamma) - \mathcal{F}(\Gamma_0) \leq D_1 h^4, \tag{98}$$

then there are $\Psi \in H^1_{mag}(Q_h)$ and $\xi \in H^1(Q_h \times \mathbb{R}^3_s)$ s.t.

symmetric functions

$$\alpha(r,X) = \alpha_*(r)\Psi(X) + \xi(r,X), \tag{99}$$

where

$$\sup_{0<h\leq h_0} \|\Psi\|^2_{H^1_{mag}(Q_h)} \leq C, \tag{100}$$

$$\frac{h^{-2}}{|Q_h|}\int_{Q_h} |\Psi(X)|^2 dX + \frac{h^{-4}}{|Q_h|}\langle \nabla_A\Psi, \nabla_A\Psi\rangle_{L^2(Q_h)}$$

This tells us that the H^1 norm of Ψ has the same scaling in h (as an upper bound) than the minimizer of the GL functional (it is a macroscopic object).

and

$$\|\xi\|_{H^1(Q_h\times\mathbb{R}^3_s)} \leq Ch^4\left(\|\Psi\|^2_{H^1_{mag}(Q_h)} + D_1\right). \tag{101}$$

Smaller in H^1 than leading order given by $\alpha_*\Psi$.

$$= \mathrm{tr}_Q[\xi^*\xi] + \mathrm{tr}_Q\left[(-i\nabla+A)\left(\xi^*\xi + \xi^*\xi\right)(-i\nabla+A)\right]$$

This norm is not scaled with h as the one above.

The proof of Theorem 3 in the case of a constant magnetic field is the main novelty in [DHM 2023a]. In [DHM 2023b] ideas from [FHSS 2012] are used to reduce the problem with general external fields to that treated in [DHM 2023a].

Afterward, one shows that for all Γ, whose Cooper pair wave function $\alpha = \Gamma_{12}$ satisfies (SS) it is possible to replace $\mathcal{F}(\Gamma)$ by the free energy of a trial state Γ_Δ with $\Delta(x,y) = 2V(x-y)\,\alpha_*(x-y)\,\psi\left(\frac{x+y}{2}\right)$, where ψ is the function in (SS). For this step we adapted in [DHM 2023a/b] the techniques from [FHSS 2012].

Let us mention also here some interesting technical ingredients.

- **Relative entropy inequality with an additional term:** the following inequality has been proved in [Lemma 1, FHSS 2012].

  For any $0 \leq \Gamma \leq 1$ and any Γ_0 of the form $\Gamma_0 = [1 + \exp(H)]^{-1}$ commuting with $\mathcal{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, we have

$$\mathcal{H}_0(\Gamma,\Gamma_0) \geq \widetilde{\mathrm{tr}}_{Q_h}\left[\frac{H}{\tanh(\frac{H}{2})}(\Gamma-\Gamma_0)^2\right] + \frac{4}{3}\mathrm{tr}_{Q_h}\left[\left(\Gamma(1-\Gamma)-\Gamma_0(1-\Gamma_0)\right)^2\right]. \tag{102}$$

Also this more precise inequality for the relative entropy can be proved with Klein's inequality after one has established the related inequality for numbers.

Decomposition of the Cooper pair wave function with entanglement:

in the proof of Theorem 4 we need to quantify the coercivity of the term

(that is, we want to know which norms related to α, ψ and ξ (see (80)) are dominated by this term)

$$(*)(B) = \int_{\mathbb{R}^3\times Q_h} \overline{\alpha(x,y)}\left[\frac{1}{2}K^B_{T,x} + \frac{1}{2}K^B_{T,y} + V(x-y)\right]\alpha(x,y)\,d(x,y). \tag{103}$$

This is $K^B_T = \dfrac{(-i\nabla+A_B)^2}{\tanh\left(\frac{(-i\nabla+A)^2}{2T}\right)}$ acting on the x-coordinate of $\alpha(x,y)$.

We highlight here again that $\alpha(x,y)=\alpha(y,x)$ holds.

Using this symmetry and Fourier analysis, the following inequality has been proved in [Lemma 3, FHSS 2012]:

$$(*)(0) \geq \text{const.}\, h^2 \int_{\mathbb{R}^3 \times Q_h} |(\nabla_x + \nabla_y)\alpha(x,y)|^2 \, d(x,y). \qquad (104)$$

This would be easy to prove if K_T were replaced by $-\Delta$.

In case of a constant magnetic field we write the Cooper pair wave function as

$$\alpha(r,X) = \alpha_*(r) \cos\left(\frac{r}{2}\cdot \Pi_X\right)\Psi(X) + \xi_0(r,X). \qquad (105)$$

Note that the relative- and the center-of-mass wave function are entangled.

$= -i\nabla_X + A_{2B}(X)$

The functions Ψ and ξ_0 are defined via the operator

$$(A\alpha)(X) = \int_{\mathbb{R}^3} \alpha_*(r) \cos\left(\frac{r}{2}\cdot \Pi_X\right)\alpha(r,X)\, dr, \qquad (106)$$

whose adjoint is given by

$$(A^*\Psi)(r,X) = \alpha_*(r)\cos\left(\frac{r}{2}\cdot\Pi_X\right)\Psi(X). \tag{107}$$

We have

$$\Psi = (AA^*)^{-1}A\alpha \quad \text{and} \quad \xi_0 = \alpha - A^*\Psi. \tag{108}$$

This decomposition of the Cooper pair wave function has been introduced in [FHL 2018] to study the Birman-Schwinger operator related to the operator

$$\underbrace{\underbrace{\frac{(-i\nabla_x + A_B(x))^2 + (-i\nabla_y + A_B(y))^2 - 2\mu}{\tanh\left(\frac{(-i\nabla_x + A_B(x))^2}{2T}\right) + \tanh\left(\frac{(-i\nabla_y + A_B(y))^2}{2T}\right)}}_{L_{T,B}} - \underbrace{V(x-y)}_{\geq 0}}_{\text{Hessian of the BCS functional at the normal state in the non-translation invariant case.}} \tag{105}$$

It can be shown that $L_{T,B} - V$ has zero as eigenvalue

If one is in an eigenvalue of the Birman-Schwinger operator $V^{1/2} L_{T,B} V^{1/2}$ ($L_{T,B} = M_{T,B}^{-1}$, this op. has been studied in [FHL 2019]). We prove one direction: assume that

$$(M_{T,B} - V)\,\alpha = 0 \tag{110}$$

holds. When we multiply the equation with $V^{1/2}$, we find

$$\underbrace{V^{1/2}\alpha}_{\phi} = V^{1/2} L_{T,B} V^{1/2} \underbrace{V^{1/2}\alpha}_{\phi} , \tag{111}$$

which proves the claim. It also shows the relation $\phi = V^{1/2}\alpha$ between the two eigenfunctions. This, in particular suggests that similar decompositions of the relevant functions can be used to study spectral properties of $M_T - V$ and $V^{1/2} L_{T,B} V^{1/2}$. However, that the same decomposition can also be used to study the quadratic form in (103) is less obvious.

In one of the main steps in the proof of Theorem 4 we show

$$\frac{(*)(B)}{|Q_h|} \geq \frac{1}{2} \left\{ \frac{1}{|Q_h|} \int_{Q_h} \overline{\Psi(X)} \left(-i\nabla_X + A_{2B}(X)\right)^2 \Psi(X) \, dX + \|\xi_0\|^2_{H^1(\mathbb{R}^3 \times Q_h)} \right\} \qquad (112)$$

$$- \text{const. } h^2 \left(\|\Psi\|_2^2 + h^2 \right),$$

which replaces (104) in our setting. This ends our discussion of the lower bound for the BCS free energy.

Topics to cover in the lecture

Lecture 1 (Mention that questions are welcome any time)

- Lecture notes: https://user.math.uzh.ch/deuchert

  See also C. Hainzl, R. Seiringer, The BCS functional of superconductivity and its mathematical properties, J. Math. Phys. 57, 021101 (2016)

- Introduction without writing formulas

- Write down Hamiltonian, introduce Gibbs state and the mathematical definition of superconductivity

- Quasi-free states via Wick theorem, explain Wick theorem only with 2-pdm

- Restrict to translation and SU(2) invariant quasi-free states, say we drop all interaction terms that depend on γ, and write down the translation invariant BCS functional.

- Argue that $\alpha \neq 0$ in the minimizer implies superconductivity.

- Introduce the normal state
- State Theorem 1

Lecture 2

- State Klein's inequality and Lemma 2
- Prove one direction in Theorem 1 with Lemma 2.

  Uniqueness of the normal state in the case $V \equiv 0$ follows from this argument, too!
- Remarks on Thm 1.
- Introduce BCS functional in the presence of periodic external fields

Lecture 3

- Introduce GL functional
- Relation between BCS and GL theory

- Literature in words.
- Write down and discuss Assumptions, Theorem 2 and Theorem 3
- Construction of the trial state
- Set up expansion for BCS free energy

Lecture 4

- Discuss mathematical tools for the resolvent expansion
- Discuss briefly the literature
- Strategy for proof of lower bound and Theorem 4
- Mention very briefly the improved relative entropy inequality
- Discuss coercivity and the decomposition of the Cooper pair wave function with entanglement.