Fourier Series and PDEs
(Math 4425)

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1. Introduction (Review of ODEs)

1.1. What is a differential equation?

1.2. Newton's equations

1.3. Reduction of a k-th order system to a 1st order system

1.4. Linear first order ODEs

1.1. What is a differential equation?

A differential equation is any equation, in which a function and its derivatives appear.

Simple examples: Let $a \in \mathbb{R}$ and $u : \mathbb{R} \to \mathbb{R}$.

- $\dot{u}(t) = a\, u(t)$ (describes exponential growth or decay)
- $\ddot{u}(t) = a\, u(t)$ (describes e.g. oscillations)

Let $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$.

- $\partial_t u(x,t) + a\, \partial_x u(x,t) = 0$ (describes transport, we will see that later)

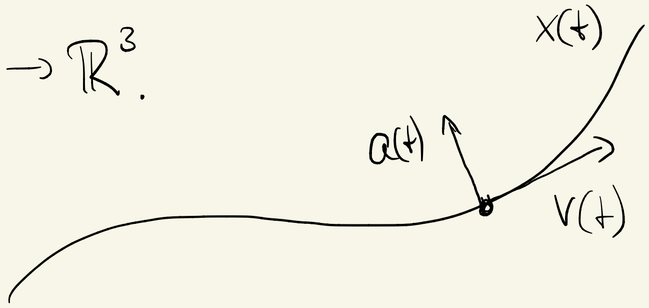
Many differential equations come from physics. This is the reason why they play such an important role in physics, chemistry, and engineering. But they also play an important role in mathematics, e.g., in analysis, differential geometry, topology, probability, ...

If we want to describe e.g. the motion of a planet, we need to solve a differential equation. We have a closer look at this situation in the next section.

1.2. Newton's equations

In classical mechanics a particle is described by a point in space whose location is given by a function

$$x : \mathbb{R} \to \mathbb{R}^3. \tag{1.1}$$



The derivative $\dot{x}(t)$ of this function with respect to time is the velocity of the particle

$$v = \dot{x} : \mathbb{R} \to \mathbb{R}^3 \tag{1.2}$$

and the derivative of the velocity is the acceleration

$$a = \dot{v} = \ddot{x} : \mathbb{R} \to \mathbb{R}^3. \tag{1.3}$$

In such a model the particle is usually moving in a force field

$$F : \mathbb{R}^3 \to \mathbb{R}^3, \tag{1.4}$$

which exerts a force $F(x)$ on the particle at x.

← fat and blue is a definition

Then **Newton's second law of motion** states that, at each point x in space, the forcing acting on the particle must be equal to the acceleration times the mass (a positive constant) of the particle, that is,

$$m\ddot{x}(t) = F(x(t)), \quad \text{for all } t \in \mathbb{R}. \qquad (1.5)$$

Equation (1.5) is an **ordinary differential equation (ODE)** because x depends only on one variable. It is of **second order** since the highest derivative of x appearing in the equation is of degree two. If F is linear/nonlinear we say that (1.5) is a **linear/nonlinear ODE**. More precisely, (1.5) is a **system**

of ordinary differential equations since there is one equation for each of the three components $x_i : \mathbb{R} \to \mathbb{R}$, $i = 1,2,3$ of x.

In our case x is called the **dependent variable** and t is called the **independent variable**. It is always possible to decrease the order of an ODE to one if we are willing to increase the number of dependent variables. This is achieved if we consider the **first order system**

fat and purple is used to highlight

$$\begin{aligned} \dot{x}(t) &= v(t) \\ \dot{v}(t) &= \frac{1}{m} F(x(t)) \end{aligned} \tag{1.6}$$

instead of (1.5). Note that we now have **two** dependent variables, x and v. The number of independent variables stays the same.

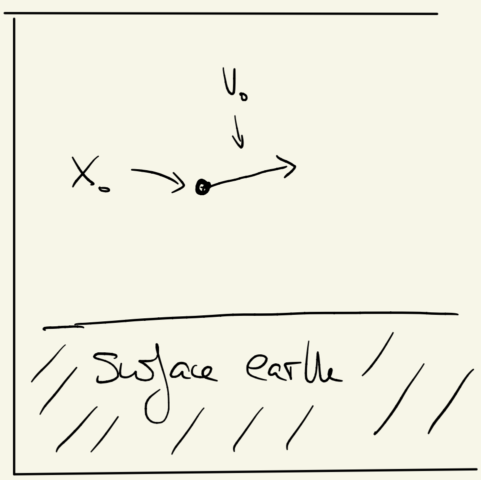
Examples: • Stone falling towards the surface of the earth. In this case the force field is approximately given by

$$F(x) = -\underset{\text{mass}}{m}\,\underset{\text{gravitational (positive constant)}}{g}\begin{pmatrix}0\\0\\1\end{pmatrix}. \qquad (1.7)$$

Hence, our system of differential equations reads

$$\begin{aligned} m\ddot{x}_1 &= 0, \\ m\ddot{x}_2 &= 0, \\ m\ddot{x}_3 &= -mg. \end{aligned} \qquad (1.8)$$

As we will see in a second, Eq. (1.8) has many solutions. To pick one, we need to fix initial conditions. Since



each equation in (1.8) is of second order, we need to fix two parameters. In our case it is natural to prescribe the position $x(0)$ and the velocity $v(0)$ of the particle at $t=0$. Let $\bar{x}, \bar{v} \in \mathbb{R}^3$ and assume that

$$x(0) = \bar{x}, \qquad v(0) = \bar{v}. \tag{1.9}$$

To solve (1.8), we integrate all three equations first from 0 to s and then again from 0 to t:

$$0 = \int_0^t \underbrace{\int_0^s m\, \ddot{x}_1(y)\, dy}\, ds = m \int_0^t \left(\dot{x}_1(s) - \bar{v}_1\right) ds$$

$$= m\left(\dot{x}_1(s) - \dot{x}_1(0)\right) \tag{1.10}$$

$$= m\left(x_1(t) - x_1(0) - \bar{v}_1 t\right) = m\left(x_1(t) - \bar{x}_1 - \bar{v}_1 t\right).$$

(same for eq. with $\ddot{x}_1$ replaced by $\ddot{x}_2$)

$$-mg \underbrace{\int_0^t \underbrace{\int_0^s dy}_{=s}\, ds}_{=\frac{1}{2}t^2} = \underbrace{\int_0^t \int_0^s m\ddot{x}_3(y)\, dy\, ds}_{= m\left(x_3(t) - \bar{x}_3 - \bar{v}_3 t\right).} \tag{1.11}$$

See previous page

We conclude that

$$\begin{aligned} x_1(t) &= \bar{x}_1 + \bar{v}_1 t, \\ x_2(t) &= \bar{x}_2 + \bar{v}_2 t, \\ x_3(t) &= \bar{x}_3 + \bar{v}_3 t - \frac{g}{2} t^2, \end{aligned} \tag{1.12}$$

or in more compact notation

$$x(t) = \bar{x} + \bar{v}t - \frac{g}{2} t^2 e_3 . \tag{1.13}$$

⁝ **Planet encircling the sun placed at the center of our coordinate system.** In this case we need to work with the gravitational force, which, unlike in the previous case, depends on x. It reads

$$F(x) = -\overset{\text{gravitational constant, positive}}{\gamma}\; \underset{\text{mass of planet}}{m}\; \underset{\text{mass of Sun}}{M}\, \frac{x}{|x|^3} \;; \qquad x \neq 0. \tag{1.14}$$

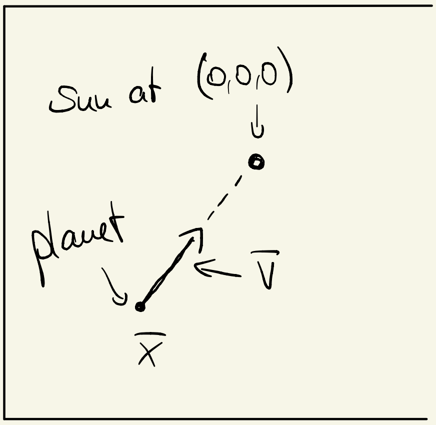
Our system of differential equations is now given by

$$m\ddot{x}_1 = -\frac{\gamma m M x_1}{\left(x_1^2 + x_2^2 + x_3^2\right)^{3/2}}$$

$$m\ddot{x}_2 = -\frac{\gamma m M x_2}{\left(x_1^2 + x_2^2 + x_3^2\right)^{3/2}}$$

$$m\ddot{x}_3 = -\frac{\gamma m M x_3}{\left(x_1^2 + x_2^2 + x_3^2\right)^{3/2}} \tag{1.15}$$

and it is no longer as easy to solve. As long as the initial conditions are such that the planet does <u>not</u> move on a straight line towards $x=0$ (see picture), the system (1.15) always has a unique solution that exists for all times. The solution can be a circle, an ellipsoid, a parabola, or a hyperbola (conic sections). We will, however, not go into more details here.



Take home message: Solutions to ODEs do <u>not</u> need to exist for all times. In our example this happens because the planet falls into the sun.

Summary: An **ordinary differential equation (ODE)** is an equation of the form

$$F\left(x(t), x^{(1)}(t), \dots, x^{(k)}(t)\right) = 0. \qquad (1.16)$$

$x : \mathbb{R} \to \mathbb{R}^n$, $n \in \mathbb{N}$

just think of $n = 1,2,3$

k-th order derivative of x

$F : \mathbb{R}^{kn} \to \mathbb{R}^n$

- The ODE is called **linear** if F is a linear function (see e.g. first two eqs. on p. 1). If F is not linear it is called **nonlinear**.
- The ODE in (1.16) is **of k-th order** since this is the highest derivative of x.
- We say (1.16) is a **system of ODEs** if $n > 1$.

- For a k-th order ODE we need to prescribe k initial conditions to be able to find a unique solution. Often this will be prescribed values for $x(0), x^{(1)}(0), \dots x^{(k-1)}(0)$.

- ODEs may fail to have solutions. If they have a solution it may not exist for all times (see the example on p. 10 and the related exercise).

- ODEs may fail to have a unique solution for prescribed initial conditions. This will be discussed further in the exercises.

- A **solution (or classical solution)** to a k-th order ODE is a C^k-function (k-times continuously differentiable) solving the equation.

- An ordinary differential equation is said to be **well-posed** if it admits a unique solution (for given initial data) and if that solution depends continuously on the parameters (as e.g. the initial conditions or parameters appearing in the equation). This is relevant because these parameters usually are not known exactly (that is without errors e.g. resulting from a measurement) in applications.

The reduction of k-th order systems to first order systems will be discussed in the next section.

1.3. Reduction of a k-th order system to a first order system

Let us consider the following system of ordinary differential equations

$$
\begin{aligned}
x_1^{(k)} &= f_1\left(t, x, x^{(1)}, \dots, x^{(k-1)}\right), \\
x_2^{(k)} &= f_2\left(t, x, x^{(2)}, \dots, x^{(k-1)}\right), \\
&\vdots \\
x_n^{(k)} &= f_n\left(t, x, x^{(2)}, \dots, x^{(k-1)}\right).
\end{aligned}
\tag{1.17}
$$

k-th derivative

$\in \mathbb{R}$ $\qquad$ $\in \mathbb{R}$ $\qquad$ $\in \mathbb{R}^n$

$f : \mathbb{R}^{kn+1} \to \mathbb{R}$

Any such **k-th** order system can be reduced to a **first order** system by changing to the new set $y = \left(x, x^{(1)}, \dots, x^{(k-1)}\right)$ of dependent variables. This yields the **new first order system**

$$\dot{y}_1 = y_2 \quad \leftarrow \text{Note that } y_i : \mathbb{R} \to \mathbb{R}^n \text{ for } i=1..k\,!$$

$$\dot{y}_2 = y_3$$

$$\vdots$$

$$\dot{y}_{k-1} = y_k$$

$$\dot{y}_k = f(t,y). \tag{1.18}$$

Equation is called **non-autonomous** because right side depends **explicitly** on t.

We can even add t to the dependent variables $z=(t,y)$, making the right hand side independent of t

$$\dot{z}_1 = 1,$$

$$\dot{z}_2 = z_3$$

$$\vdots$$

$$\dot{z}_k = z_{k+1}$$

$$\dot{z}_{k+1} = f(z). \tag{1.19}$$

Equation is called **autonomous** because the right side does not depend **explicitly** on t (it does depend **implicitly** on t via x)

If one is interested in writing a computer program to solve ODEs, it is therefore sufficient to write a solver for first order systems.

1.4. First order linear ODEs

We start by recalling the solution to the equation ($a \in \mathbb{R}$, $x : \mathbb{R} \to \mathbb{R}$)

$$\begin{cases} \dot{x}(t) = a\,x(t) \\ x(0) = \bar{x} \end{cases} \Rightarrow \quad x(t) = \bar{x}\, e^{at}, \qquad (1.20)$$

which describes exponential growth ($a>0$) or exponential decay ($a<0$). A slightly more complicated version of this equation is ($a : \mathbb{R} \to \mathbb{R}$, $x : \mathbb{R} \to \mathbb{R}$)

another notation for exponential function: $\exp(x) = e^x$

$$\begin{cases} \dot{x}(t) = a(t)\,x(t) \\ x(0) = \bar{x} \end{cases} \Rightarrow \quad x(t) = \bar{x}\, \underbrace{\exp\left(\int_0^t a(s)\, ds \right)}_{=: A(0,t)} \qquad (1.21)$$

Definition

The solution to the equation ($a, b, x : \mathbb{R} \to \mathbb{R}$)

$$\begin{cases} \dot{x}(t) = a(t)\, x(t) + b(t) \\ x(0) = \bar{x} \end{cases} \tag{1.22}$$

because of this term the equation is called **inhomogeneous**

can be expressed in terms of A and reads

$$x(t) = A(0,t)\,\bar{x} + \int_0^t A(s,t)\, g(s)\, ds. \tag{1.23}$$

I assume that you know this already from previous classes. What I really want to discuss here is the equation ($x : \mathbb{R} \to \mathbb{R}^n$, $M \in \mathbb{R}^{n\times n}$, $n > 1$)

$$\begin{cases} \dot{x}(t) = M\, x(t) \\ x(0) = \bar{x}, \end{cases} \tag{1.24}$$

that is, a vector version of (1.20). Let me highlight, that M is an $n \times n$ matrix.

In case of the similar equation (1.20) the solution was given in terms of the exponential function. Does something similar work also for (1.24)? That is, can we define the exponential function also for a matrix? The answer to both questions is yes!

Definition (Exponential function of a matrix): Let $M \in \mathbb{R}^{n \times n}$ be an $n \times n$ matrix. We define the exponential function of M by

$$\exp(M) = \sum_{n=0}^{\infty} \frac{M^n}{n!}, \qquad M^n = \underbrace{M \cdot M \cdot \ldots \cdot M}_{n\text{-times}} \tag{1.25}$$

Let us check with a formal computation that we can use (1.25) to find a solution to (1.24).

A computation is called *formal* if not every step (e.g. moving a derivative into a sum) is justified with a proof. We define $x(t) = \exp(tM)\bar{x}$ with $\bar{x} \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n \times n}$ and compute

$$\frac{d}{dt} x(t) = \frac{d}{dt} \sum_{n=0}^{\infty} \frac{t^n M^n}{n!} \bar{x}$$

$$\overset{(*)}{=} \sum_{n=0}^{\infty} \frac{d}{dt} \frac{t^n M^n}{n!} \bar{x} = \sum_{n=1}^{\infty} \frac{t^{n-1} M^n}{(n-1)!} \bar{x}$$

$$= M \sum_{n=1}^{\infty} \frac{(tM)^{n-1}}{(n-1)!} \bar{x} = M \underbrace{\sum_{n=0}^{\infty} \frac{t^n M^n}{n!}}_{\exp(tM)} \bar{x}. \quad (1.26)$$

This looks good! If the series in (1.25) converges and if we are allowed to carry out step $(*)$ in (1.26) then we have a solution to (1.24). It can be shown that these two statements are correct. We will, however, not discuss their proofs here.

Example: Assume that M is a real symmetric $n\times n$ matrix. From your linear algebra class you know that M can be diagonalized. That is, there exist n eigenvalues $\lambda_1 \dots \lambda_n$ and an orthogonal matrix O with

$$M = \underset{\substack{\uparrow \\ \text{transpose.}}}{O^t} \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} O. \qquad (1.27)$$

For the matrix exponential of M, this implies

$$\exp(M) = \sum_{j=0}^{\infty} \frac{M^j}{j!} = \sum_{j=0}^{\infty} \frac{1}{j!} \underbrace{O^t \Lambda O \, O^t \Lambda O \dots O^t \Lambda O}_{j\text{-times}}$$

$$\overset{OO^t = \mathbb{1}}{=} \sum_{j=0}^{\infty} \frac{1}{j!} O^t \begin{pmatrix} \lambda_1^j & & 0 \\ & \ddots & \\ 0 & & \lambda_n^j \end{pmatrix} O$$

$$= O^t \begin{pmatrix} \exp(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & \exp(\lambda_n) \end{pmatrix} O. \qquad (1.28)$$

That is, $\exp(M)$ is diagonal in the same basis as

M and its eigenvalues are given by $e^{\lambda_1} \dots e^{\lambda_n}$.

Having found the solution to (1.24), we can also consider the following problem ($x, b : \mathbb{R} \to \mathbb{R}^n$, $M \in \mathbb{R}^{n \times n}$):

$$\begin{cases} \dot{x}(t) = M x(t) + b(t), \\ x(0) = \bar{x}. \end{cases} \tag{1.29}$$

The solution to this equation is given by (please check!)

$$x(t) = \exp(tM)\bar{x} + \int_0^t \exp(M(t-s))\, b(s)\, ds. \tag{1.30}$$

What happens if the matrix M is time-dependent? Comparison with (1.21) suggest a solution of the form ($b(t) = 0$ for all t)

" $$x(t) = \exp\left(\int_0^t M(s)\,ds\right)\bar{x}$$ "

However, this is **not** correct. The reason is that the matrices $M(t_1)$ and $M(t_2)$ for $t_1 \neq t_2$ may not commute. If one carefully analyzes this problem one sees that the solution is given by (again $b(t) = 0$ for all t)

$$OE(M)(t)\,\bar{x} = \sum_{n=0}^{\infty} \int_0^{t_1} dt_1' \int_0^{t_1'} dt_2' \cdots \int_0^{t_{n-1}'} dt_n' \; M(t_1') \cdots M(t_n')\,\bar{x}. \qquad (1.31)$$

The function $OE(M)(t)$ of M is called the **time-ordered exponential**. It satisfies

$$\begin{cases} \frac{d}{dt} OE(M)(t) = M(t)\, OE(M)(t) \\ OE(M)(t) = \mathbb{1}_{n \times n} \end{cases} \qquad (1.32)$$

We will not go into more details here.

1.5. First order nonlinear equations

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and consider the equation $(x : \mathbb{R} \to \mathbb{R}, \ \bar{x} \in \mathbb{R})$

$$\begin{cases} \dot{x}(t) = f(x(t)), \\ x(0) = \bar{x}. \end{cases} \tag{1.33}$$

If $f(x) \neq 0$ for all x we can rewrite the above equation, integrate both sides from 0 to t and find

$$\int_0^t \frac{\dot{x}(s)}{f(x(s))} \, ds = t \tag{1.34}$$

That is, any solution to (1.33) must also satisfy (1.34).

We also have

$$\int_0^t \frac{\dot{x}(s)}{f(x(s))}\,ds \underset{\uparrow}{=} \int_{x(0)}^{x(t)} \frac{1}{f(y)}\,dy =: F(x(t)). \qquad (1.35)$$

change coordinates in integral
$y = x(s)$, $dy = \dot{x}(s)\,ds$

If the function $F(x)$ is invertible then

$$x(t) = F^{-1}(t) \qquad (1.36)$$

inverse of F

is a solution to (1.34), and therefore also to (1.33).

<u>Examples:</u> • $f(x) = x^2$, $\bar{x} > 0$. Hence,

$$F(x) = \int_{\bar{x}}^{x} \frac{1}{y^2}\,dy = -\left[\frac{1}{y}\right]_{\bar{x}}^{x} = \left(\frac{1}{\bar{x}} - \frac{1}{x}\right) \qquad (1.37)$$

and

$$F(x(t)) = t \iff \frac{1}{\bar{x}} - \frac{1}{x(t)} = t$$

$$\Longleftrightarrow \qquad x(t) = \frac{\bar{x}}{1 - t\bar{x}} \, . \qquad (1.38)$$

The solution exists only for $t \in [0, \bar{x}^{-1})$. At $t = \bar{x}^{-1}$ it blows up.