Fourier Series and PDEs (Math 4425)

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2. Four important PDEs and the classification of linear second order PDEs

2.1. The transport equation

2.2. The heat equation

2.3. Laplace and Poisson equations

2.4. The wave equation

2.5. Classification of linear second order PDEs

2.1. The transport equation

Probably the simplest of all PDEs is the **transport equation** with constant coefficients. This is the equation

$$\partial_t u(x,t) + b\cdot D u(x,t) = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^n. \tag{2.1}$$

Here $u: \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$, $t \in \mathbb{R}_+$; $x, b \in \mathbb{R}^n$, and

$$Du(x,t) = \sum_{j=1}^{n} \partial_{x_j} u(x,t) e_j \tag{2.2}$$

denotes the gradient of the function u.

How do solutions to (2.1) look like? Let us write (2.1) in the form

$$\tilde{b}\cdot\tilde{D}\, u(x,t) = 0 \quad \text{with } \tilde{b} = (b,1) \text{ and } \tilde{D} = (D, \partial_t). \tag{2.3}$$

That is, the directional derivative of $u(x,t)$ in the direction of $\tilde{b}$ vanishes. In other words, in the direction of $\tilde{b}$ the function $u(x,t)$ is constant!

This motivates the following computation: we define

$$z(s) = u(x+sb,\, t+s)\,, \quad s \in \mathbb{R}, \tag{2.4}$$

where u denotes a solution to (2.1). Then we compute

$$\frac{d}{ds} z(s) = b \cdot Du(x+sb,\, t+s) + \partial_t u(x+sb,\, t+s) \tag{2.5}$$

$$\underset{\substack{\uparrow \\ \text{use (2.1)}}}{=} 0.$$

Thus $z(s)$ is a constant function of s, and consequently for each point (x,t), u is constant on the line through (x,t) with direction $(b,1) \in \mathbb{R}^{n+1}$. Hence, if we know the value of u at any point on each such line, we know its value everywhere.

2.1.1. Initial-value problem

This motivates the formulation of the following initial-

Value problem

$\mathbb{R}_+ \overset{!!}{=} (0,\infty)$ ← Definition

$$
\begin{cases}
\partial_t u(x,t) + b \cdot Du(x,t) = 0 & \text{in } \mathbb{R}^n \times \mathbb{R}_+ \\
u(x,0) = g(x) & \text{on } \mathbb{R}^n \times \{t=0\}
\end{cases}
\tag{2.6}
$$

Here $b \in \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}$ are known, and the problem is to find u.

Given (x,t), the line through (x,t) with direction $(1,b)$ equals the set $\{(x+sb, t+s) \mid s \in \mathbb{R}\}$. This line hits the plane $\Gamma = \mathbb{R}^n \times \{t=0\}$ when $s=-t$, at the point $(x-tb, 0)$. Since u is constant on this line and $u(x-tb,0) = g(x-tb)$, we deduce

$$
u(x,t) = g(x-tb). \tag{2.7}
$$

As one can easily check, this choice for u indeed solves (2.6). To make sense of this reasoning we require $g \in C^1(\mathbb{R}^n)$, that is, g is continuously differentiable.

2.1.2. The inhomogeneous problem

Next we look at the associated nonhomogeneous problem

$$\begin{cases} \partial_t u(x,t) + b\cdot Du(x,t) = f(x,t) & \text{in } \mathbb{R}^n \times \mathbb{R}_+ \\ u(0,x) = g(x) & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases} \tag{2.8}$$

Let's quickly recall how we solved the equation

$$\begin{cases} \dot{x}(t) = a(t)x(t) + b(t), \\ x(0) = \bar{x}. \end{cases} \qquad \left(\begin{array}{l} x, a, b : \mathbb{R}_+ \to \mathbb{R} \\ \bar{x} \in \mathbb{R} \end{array} \right) \tag{2.9}$$

We first solved the homogeneous problem

$$\begin{cases} \dot{x}(t) = a(t)x(t) \\ x(0) = \bar{x} \end{cases} \Rightarrow \quad x(t) = \underbrace{\exp\left(\int_0^t a(s)\,ds\right)}_{\text{Propagator } A(0,t)} \bar{x} \tag{2.10}$$

and then wrote the solution to the inhomogeneous problem in terms of the propagator as

$$x(t) = A(0,t)\bar{x} + \int_0^t A(s,t)\, b(s)\, ds. \tag{2.11}$$

In analogy, we define the **Propagator $\mathcal{P}$ of the transport equation** by

$$\mathcal{P}(s,t)\, g(x) = g\big(x - (t-s)b\big). \tag{2.12}$$

The natural candidate for the solution to (2.8) then reads

$$u(x,t) = \mathcal{P}(0,t)\, g(x) + \int_0^t \mathcal{P}(s,t)\, f(x,s)\, ds$$

$\mathcal{P}$ shifts the x-coordinate and leaves s unchanged.

$$= g(x-tb) + \int_0^t f\big(x-(t-s)b, s\big)\, ds. \tag{2.13}$$

Let's check:

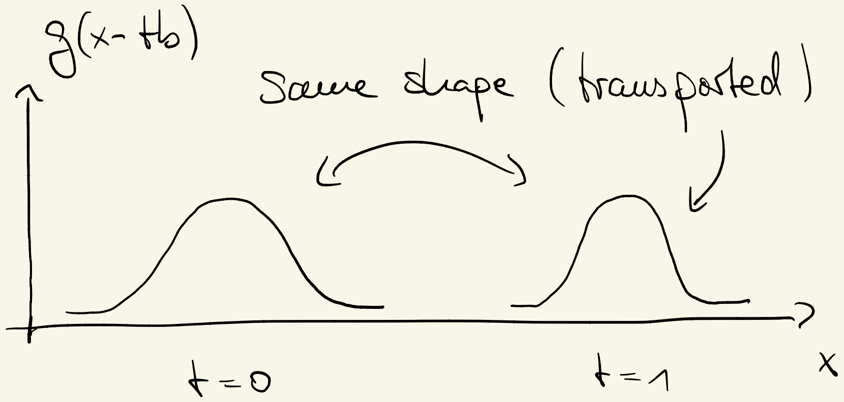
$$\partial_t u(t,x) = \partial_t g(x-tb) + \partial_t \int_0^t f\big(x-(t-s)b, s\big)\, ds$$

$$= -b\cdot Dg(x-tb) + f(x,t) - b\int_0^t Df\big(x-(t-s)b, s\big)\, ds$$

$$= -b\cdot Du(x,t) + f(x,t). \tag{2.14}$$

Moreover, $u(x,0) = g(x)$. We conclude that the function $u(x,t)$ defined in (2.13) indeed solves (2.8)! For all steps to make sense we require $g \in C^1(\mathbb{R}^n, \mathbb{R})$ and that $f: \mathbb{R}^n \times \mathbb{R}_+$ is a continuous function that is additionally continuously differentiable with respect to the first argument (the space coordinate). Note that we did not differentiate f with respect to time.

Here is a picture how solutions to the homogeneous transport equation look like for $n=1$.



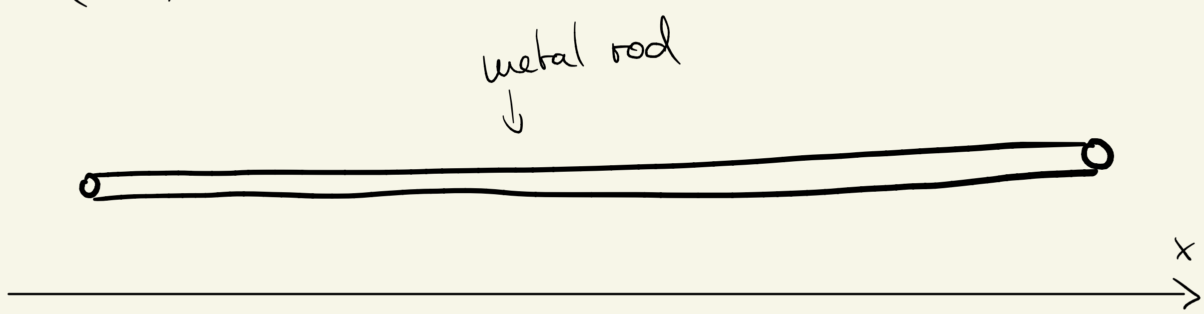
This ends our discussion of the transport equation.

2.2. The heat equation

In this section we discuss the problem of heat diffusion. We start with a derivation of the heat equation from basic physical principles. For the transport equation we omitted this step because the equation is so simple.

2.2.1. Derivation of the heat equation

Let us consider an infinite metal rod and suppose we are given an initial heat distribution at time $t=0$. Let the temperature at the point $x\in\mathbb{R}$ at time $t\in\mathbb{R}_+=(0,\infty)$ be denoted by $u(x,t)$.



For a small number $h>0$ and some $x_0 \in \mathbb{R}$ consider now the interval $S = [x_0, x_0+h]$ of length h.

The amount of heat energy in S at time t is given by

$$H(t) = \sigma \int_S u(x,t)\,dx, \tag{2.1}$$

where $\sigma > 0$ is a constant called the specific heat of the material. Therefore, the heat flow into S is

$$\partial_t H(t) = \sigma \int_S \partial_t u(x,t)\,dx, \tag{2.2}$$

which is approximately equal to

$$\sigma h\, \partial_t u(x_0,t) \tag{2.3}$$

since the length of S is h (we assume here that $\partial_t u(x,t)$ is a continuous functions of x).

Now we apply Newton's law of cooling, which states that heat flows from the higher to lower temperature at a rate proportional to the temperature difference, that is, the derivative. The heat flow through the right end of our interval is therefore

$$k\,\partial_x u(x_0+h,t) \tag{2.4}$$

where $k>0$ is the heat conductivity of the material. A similar argument for the other side shows that the total heat flow through S is given by

$$k\left[\partial_x u(x_0+h,t) - \partial_x u(x_0,t)\right], \tag{2.5}$$

where $k>0$ is the thermal conductivity of the material. We therefore have

$$bh\,\partial_t u(x_0,t) = k\left[\partial_x u(x_0+h,t) - \partial_x u(x_0,t)\right]$$

$$\Leftrightarrow \quad \frac{b}{k}\partial_t u(x_0,t) = \frac{\partial_x u(x_0+h,t) - \partial_x u(x_0,t)}{h}$$

$$\xrightarrow{h\to 0} \partial_x^2 u(x_0,t). \tag{2.6}$$

In the limit $h \to 0$ we thus discovered the **heat equation**

$$\partial_t u(x,t) = \partial_x^2 u(x,t). \tag{2.7}$$

A similar derivation can be carried out if one replaces $x \in \mathbb{R}$ by $x \in \mathbb{R}^n$ and one finds

$$\partial_t u(x,t) = \Delta u(x,t) \tag{2.8}$$

with the **Laplace operator** or **Laplacian** Δ defined by

$$\Delta u(x,t) = \sum_{j=1}^{n} \partial_{x_j}^2 u(x,t). \tag{2.9}$$

One is often interested in solving the **initial value problem**

$$\begin{cases} \partial_t u(x,t) = \Delta u(x,t) & \text{in } \mathbb{R}^n \times \mathbb{R}_+, \\ u(x,0) = g(x) & \text{in } \mathbb{R}^n \times \{t=0\}, \end{cases} \tag{2.10}$$

with a continuous function g that goes to zero for $|x| \to \infty$. In the presence of **heat sources**

Additional example (Intermezzo)

Let us consider the one-dimention heat equation

$$\partial_t u(x,t) = \partial_x^2 u(x,t) \quad \text{in } \mathbb{R}\times\mathbb{R}_+ . \tag{$*$}$$

Claim: The function $u(x,t) = \frac{1}{\sqrt{t}} \exp\left(-\frac{x^2}{4t}\right)$ solves $(*)$.

Let's check: $\cdot$ $\partial_t u(x,t) = \partial_t \frac{1}{\sqrt{t}} \exp\left(-\frac{x^2}{4t}\right)$

$$= -\frac{1}{2} t^{-3/2} \exp\left(-\frac{x^2}{4t}\right) + \frac{1}{\sqrt{t}} \exp\left(-\frac{x^2}{4t}\right) \frac{x^2}{4t^2}$$

$\cdot$ $\partial_x u(x,t) = \partial_x \frac{1}{\sqrt{t}} \exp\left(-\frac{x^2}{4t}\right) = \frac{1}{\sqrt{t}} \exp\left(-\frac{x^2}{4t}\right)\left(-\frac{x}{2t}\right)$

$\cdot$ $\partial_x^2 u(x,t) = \partial_x \frac{1}{\sqrt{t}} \exp\left(-\frac{x^2}{4t}\right)\left(-\frac{x}{2t}\right)$

$$= \frac{1}{\sqrt{t}} \exp\left(-\frac{x^2}{4t}\right)\left(\frac{x}{2t}\right)^2 + \frac{1}{\sqrt{t}} \exp\left(-\frac{x^2}{4t}\right)\left(-\frac{1}{2t}\right)$$

Let's insert this into $(*)$ and check. We find

$$-\frac{1}{2}t^{-3/2}\exp\left(-\frac{x^2}{4t}\right)+\frac{1}{\sqrt{t}}\exp\left(-\frac{x^2}{4t}\right)\frac{x^2}{4t^2}$$

$$=\frac{1}{\sqrt{t}}\exp\left(-\frac{x^2}{4t}\right)\left(\frac{x}{2t}\right)^2+\frac{1}{\sqrt{t}}\exp\left(-\frac{x^2}{4t}\right)\left(-\frac{1}{2t}\right).$$

We conclude that $u(x,t)=\frac{1}{\sqrt{t}}\exp\left(-\frac{x^2}{4t}\right)$ indeed solves $(*)$. How does it look like? It is a Gaussian function that is getting broader and broader for larger t. Moreover, we have

$$\int_{-\infty}^{\infty}u(x,t)\,dx=\frac{1}{\sqrt{t}}\int_{-\infty}^{\infty}\exp\left(-\frac{x^2}{4t}\right)dx$$

$$\overset{\substack{y=\frac{x}{\sqrt{t}}\\ dy=\frac{dx}{\sqrt{t}}}}{=}\int_{-\infty}^{\infty}\exp\left(-\frac{y^2}{4}\right)dy.$$

That is, the integral $\int_{-\infty}^{\infty}u(x,t)\,dx$ does **not** depend on t as it should be (energy conservation, see derivation of heat equation).

$\square$

described by a continuous function $f(x,t)$ one is interested in the **inhomogeneous problem**

$$\begin{cases} \partial_t u(x,t) - \Delta u(x,t) = f(x,t) & \text{in } \mathbb{R}^n \times \mathbb{R}_+, \\ u(x,0) = g(x) & \text{in } \mathbb{R}^n \times \{t=0\}. \end{cases} \qquad (2.11)$$

The heat equation can also be studied in an interval of finite length or, more generally, in a subset of $\mathbb{R}^n$ with finite volume (as e.g. the unit disc $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\} \subset \mathbb{R}^2$). As in the case of the problem on $\mathbb{R}^n$ we need to prescribe an initial condition g, and we need to prescribe boundary condition (In (2.10) and (2.11) we assumed that $g(x) \to 0$ for $|x| \to \infty$, which then also holds for the solution $u(x,t)$. That is, we "set a boundary condition at $|x| = \infty$".)

In the case $n=1$ (metallic rod) the most common boundary conditions are

BC = boundary condition

(a) Prescribed temperature (Dirichlet BCs)

We have
$$u(0,t) = u_\ell \in \mathbb{R},$$
$$u(L,t) = u_r \in \mathbb{R}, \quad (L = \text{length of rod})$$

or
$$u(0,t) = u_\ell(t),$$
$$u(L,t) = u_r(t). \tag{2.12}$$

(b) Prescribed temperature flux (Neumann BCs)

We have
$$\frac{\partial u}{\partial x}(0,t) = \phi_\ell(t),$$
$$\frac{\partial u}{\partial x}(L,t) = \phi_r(t). \tag{2.13}$$

In case of insulating boundaries, i.e. zero flux, these would be

$$\frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(L,t). \tag{2.14}$$

(c) Radiating (Robin) boundary conditions

In the case where one or both ends are such that the heat flux is proportional to the temperature, we have

$$\frac{\partial u}{\partial x}(0,t) = \underbrace{k_1\, u(0,t)\,, \qquad \frac{\partial u}{\partial x}(L,t) = k_2}_{\in\mathbb{R},\ \text{could also be time-dependent}}\, u(L,t). \qquad (2.15)$$

Initial condition: We need to prescribe an initial condition that satisfies the boundary conditions.

Remarks: In higher space dimensions we need to prescribe the value of the function at the boundary or its normal derivative (directional derivative pointing outward) or a relation between them. An example will be discussed in the next section.

Examples: • Heat equation with Dirichlet boundary conditions

Let us consider the initial value problem

$$\begin{cases} \partial_t u(x,t) = \partial_x^2 u(x,t) & \text{in } [-\pi/2, \pi/2] \times \mathbb{R}_+, \\ u(x,t) = 0 & \text{on } \{-\pi/2, \pi/2\} \times \mathbb{R}_+, \\ u(x,0) = \cos(x) & \text{on } [-\pi/2, \pi/2] \times \{t=0\}. \end{cases}$$

Claim: The solution is given by

$$u(x,t) = e^{-t}\cos(x).$$

Check: (a) $\partial_t u(x,t) = -e^{-t}\cos(x)$

(b) $\partial_x^2 u(x,t) = e^{-t}\partial_x\big(-\sin(x)\big) = -e^{-t}\cos(x)$

(c) $u(-\pi/2, t) = 0 = u(\pi/2, t)$

(d) $u(x,0) = \cos(x)$

Note: $\lim_{t\to\infty} u(x,t) = 0$ for all $x \in [-\pi/2, \pi/2]$.

∴ Heat equation with Neumann boundary conditions

Now we consider the initial value problem

$$\begin{cases} \partial_t u(x,t) = \partial_x^2 u(x,t) & \text{in } [-\pi/2, \pi/2] \times \mathbb{R}_+ , \\ \partial_x u(x,t) = 0 & \text{on } \{-\pi/2, \pi/2\} \times \mathbb{R}_+ , \\ u(x,0) = 1 & \text{on } [-\pi/2, \pi/2] \times \{t=0\} . \end{cases}$$

Claim: The solution reads $u(x,t) = 1$.

Check:

$$\partial_t u(x,t) = 0 = \partial_x^2 u(x,t)$$

$$\partial_x u(-\pi/2, t) = 0 = \partial_x u(\pi/2, t)$$

$$u(x,0) = 1$$

Note: $u(x,t) = 1$ is a steady state of the above initial value problem. That is, it does not depend on time.

2.3. Laplace and Poisson equations

Assume we consider the heat equation in the two-dimensional **unit disc**

$$D = \{ (x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1 \} \tag{2.16}$$

with Dirichlet boundary conditions, i.e.,

$$\begin{cases} \partial_t u(x,t) = \Delta u(x,t) & \text{in } D \times \mathbb{R}_+, \\ u(x,0) = g(x) & \text{in } D \times \{t=0\}, \\ u(x,t) = f(x) & \text{on } \partial D. \end{cases} \tag{2.17}$$

Here

$$\partial D = \{ (x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1 \} \tag{2.18}$$

denotes the **unit circle**, which happens to be the boundary ∂D. The function $f: D \to \mathbb{R}$ is assumed to satisfy $g(x) = f(x)$ for all $x \in \partial D$.

After a long period of time, there will be no more heat exchange, so that the system reaches thermal equilibrium and $\partial_t u = 0$. In this case, the heat equation reduces to the **Laplace equation**

$$\begin{cases} \Delta u(x) = 0 & \text{in } D, \\ u(x) = g(x) & \text{on } \partial D. \end{cases} \tag{2.19}$$

Solutions to the equation $\Delta u = 0$ are called **harmonic functions** and have many interesting properties (we will discuss this later).

The steady state solutions of the **inhomogeneous heat equation**

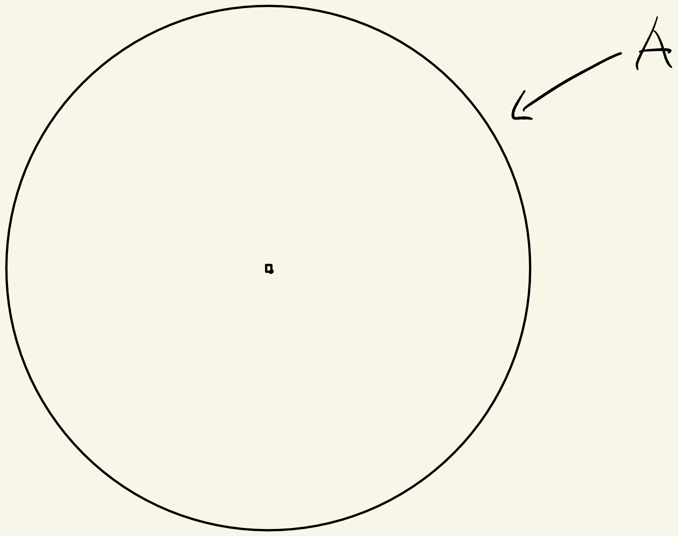
$$\begin{cases} \partial_t u(x,t) - \Delta u(x,t) = g(x) & \text{in } D \times \mathbb{R}_+, \\ u(x,0) = g(x) & \text{in } D \times \{t=0\}, \\ u(x,t) = f(x) & \text{on } \partial D, \end{cases} \tag{2.20}$$

with $\varrho : D \to \mathbb{R}$ satisfy the **Poisson equation**

$$\begin{cases} \Delta u(x) = \varrho(x) & \text{in } D, \\ u(x) = g(x) & \text{on } \partial D. \end{cases} \tag{2.21}$$

Example

Let us consider the Laplace equation $\Delta u(x) = 0$ in the annulus $A = \{ x \in \mathbb{R}^2 \mid 0 < \|x\|_2 < 1 \}$. As



boundary condition we impose $u(x) = 0$ for all x with $\|x\|_2 = 1$ and $u(0) = +\infty$. The second condition may seem funny at first sight but it will allow us to find a beautiful solution that is also a steady state of the heat equation.

Claim: The function $u(x) = -\ln(\|x\|_2)$ solves the Laplace equation with the prescribed boundary conditions.

Let's check: $\boxdot$ $\partial_{x_1} \ln(\underbrace{\|x\|_2}_{\sqrt{x_1^2+x_2^2}}) = \frac{x_1}{\|x\|_2^2}$

$\boxdot$ $\partial_{x_1}^2 \ln(\|x\|_2) = \frac{1}{\|x\|_2^2} - \frac{2x_1^2}{\|x\|_2^2}$

By symmetry, we have

$\boxdot$ $\partial_{x_2}^2 \ln(\|x\|_2) = \frac{1}{\|x\|_2^2} - \frac{2x_2^2}{\|x\|_2^2}$.

We conclude that

$$\Delta u(x) = \partial_{x_1}^2 u(x) + \partial_{x_2}^2 u(x) = -\frac{2}{\|x\|_2^2} + \frac{2\overbrace{(x_1^2+x_2^2)}^{=\|x\|_2^2}}{\|x\|_2^4}$$

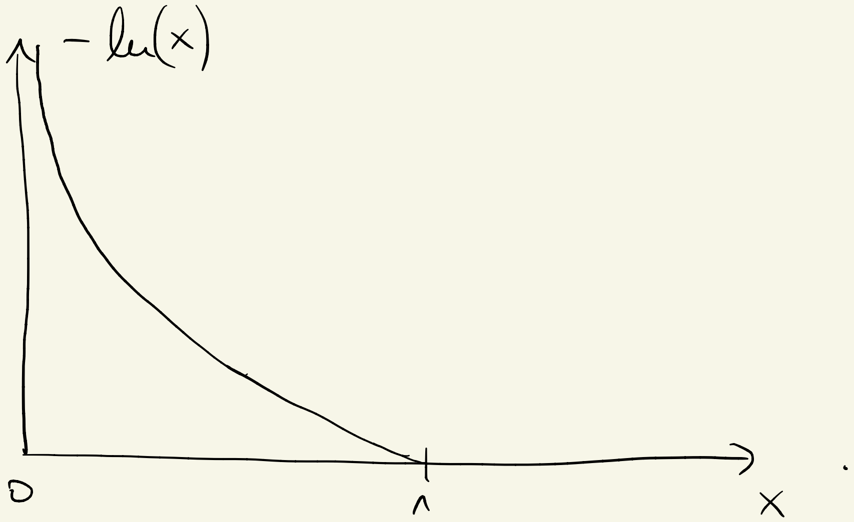
$$= 0.$$

$\boxdot$ Let x s.t. $\|x\|_2 = 1$. Then $u(x) = -\ln(1) = 0$.

$\boxdot$ We have $\lim_{\|x\|_2 \to 0} u(x) = \lim_{\|x\|_2 \to 0} -\ln(\|x\|_2) = +\infty$.

Interpretation as steady state of heat eq.: We

have a strong pointwise source of heat at $x=0$ at temperature $+\infty$, which pumps the same amount of heat into the system that is lost at the other boundary $\left(\{x\in\mathbb{R}^2 \mid \|x\|_2=1\}\right)$, which has temperature 0.

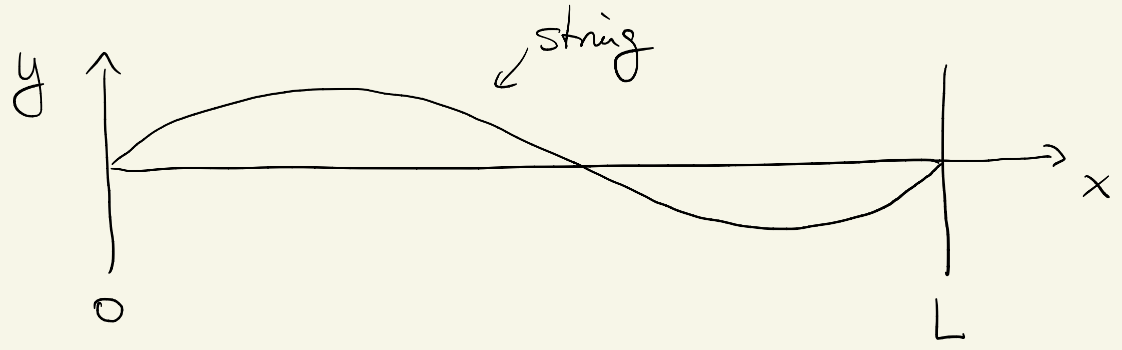


2.4. The wave equation

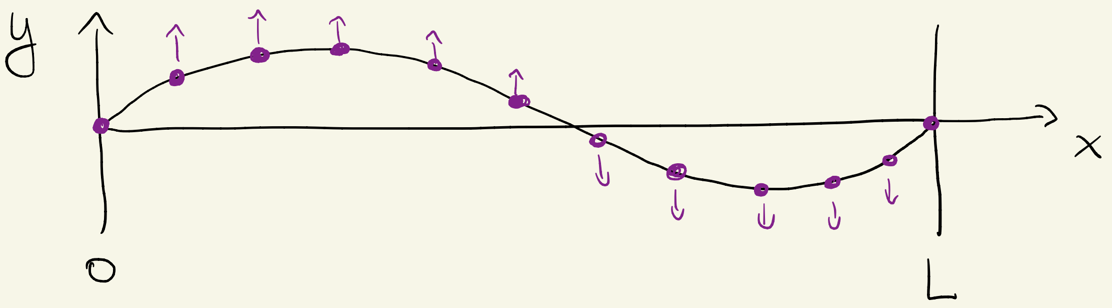
Our first goal in this section is to derive the equation of motion for a vibrating string. A very good example are the strings in a piano.

2.4.1. Derivation of the wave equation

Imagine a string placed in the (x,y)-plane, and stretched along the x-axis between $x=0$ and $x=L$. If it is set to vibrate, its displacement $y = u(x,t)$ is then a function of x and t.



To derive an equation of motion for the string, we consider it as being subdivided into a large number N of masses (which we think of as individual particles) distributed along the x-axis.



Like this, the n-th particle has its x-coordinate at $x_n = \frac{nL}{N}$. We assume that each of these particles is oscillating in the y-direction only. Moreover, each particle will have its oscillation linked to its immediate neighbor by the tension of the string.

We set $y_n(t) = u(x_n,t)$, and note that $x_{n+1} - x_n = h = \frac{L}{N}$. If we assume that the string has constant density

$\varrho > 0$, it is reasonable to assign mass equal to ϱh to each particle. By Newton's law (see first lecture) $\varrho h \ddot{y}_n(t)$ equals the force acting on the n-th particle. We now make two assumptions:

(1) Only nearest neighbor particles interact with each other.

(2) The force coming from the right of the n-th particle is given by

$$\frac{\tau}{h}\left(y_{n+1} - y_n\right), \tag{2.22}$$

where $\tau > 0$ is the coefficient of tension of the string. There is a similar force coming from the left, and it is

$$\frac{\tau}{h}\left(y_{n-1} - y_n\right). \tag{2.23}$$

We thus obtain the following equation of motion for the n-th particle:

$$\rho h \, \ddot{y}_n(t) = \frac{\tau}{h}\Big[y_{n+1}(t) + y_{n-1}(t) - 2y_n(t) \Big]$$

$$= \frac{\tau}{h}\Big[u(x_n+h,t) + u(x_n-h,t) - 2u(x_n,t) \Big]$$

Taylor approximation

$$= u(x_n,t) + \partial_x u(x_n,t)\, h + \frac{1}{2}\partial_x^2 u(x_n,t)\, h^2 + o(h^2)$$

$$\lim_{h\to 0} \frac{o(h^2)}{h^2} = 0$$

$$= \frac{\tau}{h}\Big[\partial_x^2 u(x_n,t)\, h^2 + o(h^2) \Big]$$

$$\Leftrightarrow \quad \rho\, \partial_t^2 u(x_n,t) = \frac{\tau}{h^2}\Big[\partial_x^2 u(x_n,t) + o(h^2) \Big]. \tag{2.24}$$

When we take the limit $h \to 0$ on both sides (note that this implies $N \to \infty$) we find the **wave equation**

$$\partial_t^2 u(x,t) = \underbrace{\frac{\tau}{\rho}}_{:=c} \partial_x^2 u(x,t). \tag{2.25}$$

For reasons that will be apparent later, the parameter $c>0$ is called the **velocity** of the motion.

In higher dimension the wave equation reads

$$\partial_t^2 u(x,t) = c^2 \Delta_x u(x,t). \tag{2.26}$$

The two-dimensional wave equation, e.g., describes a vibrating membrane (think of a tambourine). It is also possible to include an additional force $F(x,t)$ acting on the string or the membrane from above or below. In this case the wave equation reads

$$\frac{1}{c^2} \partial_t^2 u(x,t) - \Delta u(x,t) = F(x,t). \tag{2.27}$$

Boundary and initial conditions

As in the case of the heat equation we need to prescribe the values of $u(x,t)$ at the spatial boundary of our set. We can choose the same boundary conditions as described there. Of course, the physical interpretation of the boundary conditions changes. Dirichlet BCs describe a string that is fixed at the end points, Neumann BCs can be used to describe a string that can move freely at the boundary points, and Robin BCs can be used to model a force that pulls the string back to displacement zero and that depends linearly on the current displacement.

Similarly as in the case of the heat equation we also need to impose initial conditions. However, unlike as in this case, we must prescribe two initial conditions because the wave equation has two time derivatives. In many situations one prescribes the position and the velocity of the function $u(x,t)$ at $t=0$, that is,

$$
\begin{aligned}
u(x,0) &= f(x), \qquad \text{(position)} \\
(\partial_t u)(x,0) &= g(x). \qquad \text{(velocity)}
\end{aligned}
\tag{2.28}
$$

Examples: ◦] Wave equation in $\mathbb{R}$

Let us consider the initial value problem

$$\begin{cases} \partial_t^2 u(x,t) = c^2 \partial_x^2 u(x,t) & \text{in } \mathbb{R}\times\mathbb{R}_+, \\ u(x,0) = g(x) & \text{on } \mathbb{R}\times\{t=0\}, \\ (\partial_t u)(x,0) = -c\,\partial_x g(x) & \text{on } \mathbb{R}\times\{t=0\}, \end{cases}$$

with a function $g \in \mathcal{C}^2$.

Claim: The solution is given by $u(x,t) = g(x-ct)$.

Check:

$$\partial_x^2 u(x,t) = \partial_x^2 g(x-ct) = g''(x-ct)$$

$$\partial_t^2 u(x,t) = \partial_t^2 g(x-ct) = c^2 \underset{\substack{\uparrow \\ \text{second derivative}}}{g''}(x-ct)$$

$$u(x,0) = g(x)$$

$$\partial_t u(x,0) = \partial_t g(x-ct)\Big|_{t=0} = -c(\nabla g)(x)$$

Conclusion: Solution has the same form as that of the transport equation!

• Wave equation with Dirichlet boundary conditions

Next, we consider the equation

$$
\begin{aligned}
\partial_t^2 u(x,t) &= \partial_x^2 u(x,t) && \text{in } [-\pi/2, \pi/2] \times \mathbb{R}_+, \\
u(x,t) &= 0 && \text{on } \{-\pi/2, \pi/2\} \times \mathbb{R}_+, \\
u(x,0) &= \cos(x) && \text{on } [-\pi/2, \pi/2] \times \{t=0\} \\
(\partial_t u)(x,0) &= 0 && \text{on } [-\pi/2, \pi/2] \times \{t=0\}
\end{aligned}
$$

Claim: The solution is given by

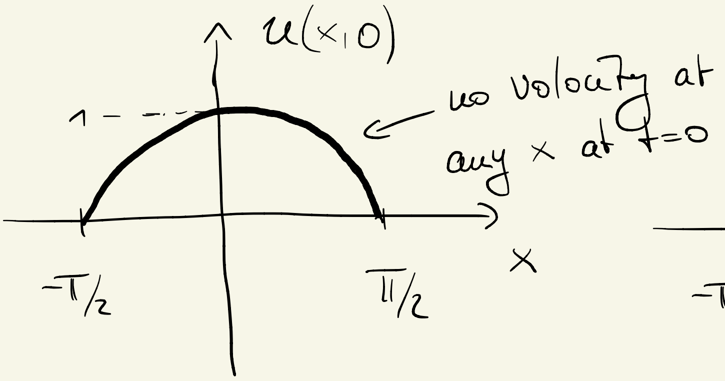
$$u(x,t) = \cos(x)\cos(t)$$

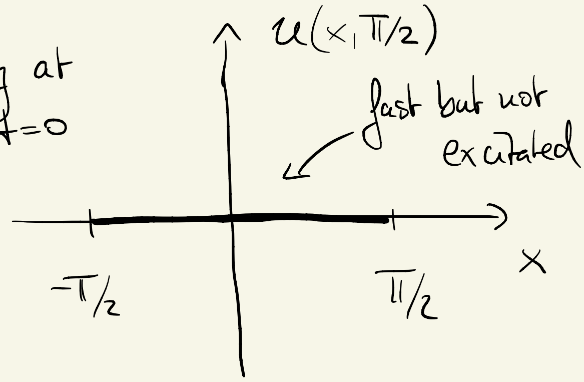
Check:

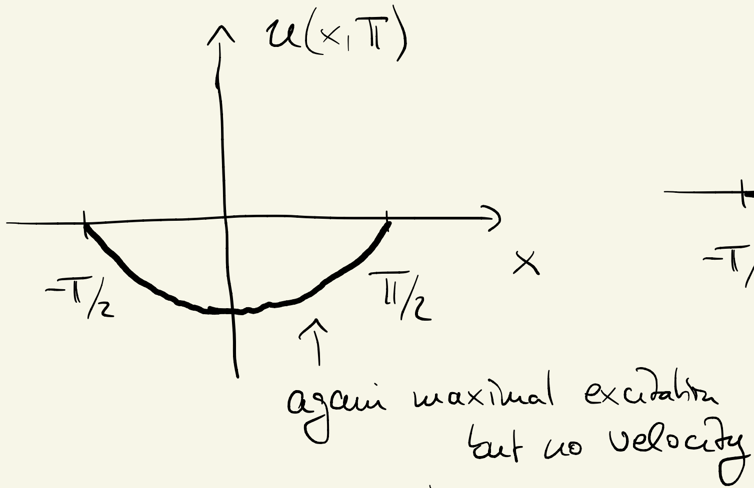
$$
\begin{aligned}
\partial_t^2 u(x,t) &= -\cos(x)\cos(t) \\
\partial_x^2 u(x,t) &= -\cos(x)\cos(t) \\
u(-\pi/2, t) &= \cos(-\pi/2)\cos(t) = 0 \\
u(\pi/2, t) &= \cos(\pi/2)\cos(t) = 0 \\
u(x,0) &= \cos(x) \; ; \; (\partial_t u)(x,0) = 0
\end{aligned}
$$

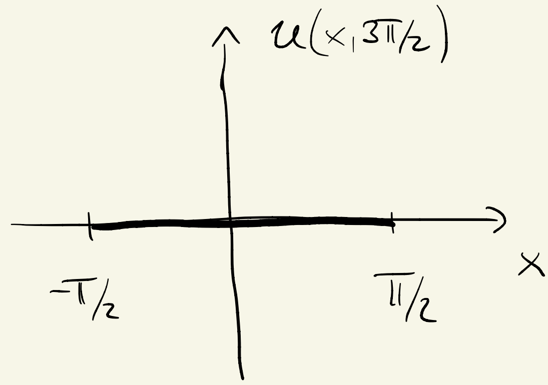
Conclusion: Shape of spatial part stays the same. Oscillation in time. This describes a

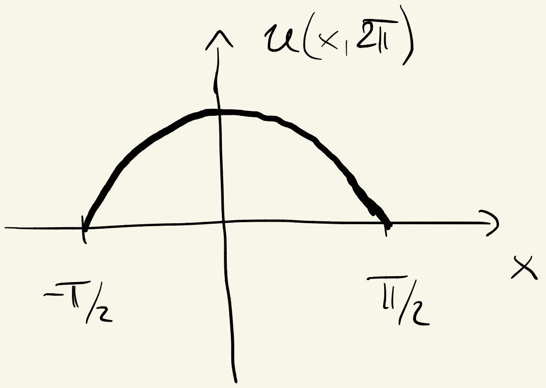
Vibrating string fixed at $-\pi/2$ and $\pi/2$.











One period of the motion of the string.

Wave equation with Neumann boundary conditions

Let's consider the equation

$$\begin{cases} \partial_t^2 u(x,t) = \partial_x^2 u(x,t) & \text{in } [-\pi/2, \pi/2] \times \mathbb{R}_+ \\ (\partial_x u)(x,t) = 0 & \text{on } \{-\pi/2, \pi/2\} \times \mathbb{R}_+ \\ u(x,0) = \sin(x) & \text{on } [-\pi/2, \pi/2] \times \{t=0\} \\ (\partial_t u)(x,0) = 0 & \text{on } [-\pi/2, \pi/2] \times \{t=0\} \end{cases}$$

Claim: The solution reads $u(x,t) = \sin(x)\cos(t)$.

Check:

$$\partial_t^2 u(x,t) = -\sin(x)\cos(t)$$

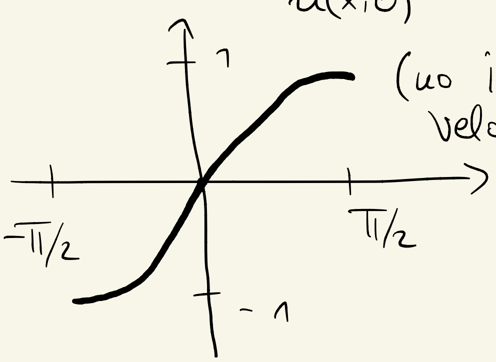
$$\partial_x^2 u(x,t) = -\sin(x)\cos(t)$$

$$\partial_x u(\pm\pi/2, t) = \cos(\pm\pi/2)\cos(t) = 0$$

$$u(x,0) = \sin(x)$$

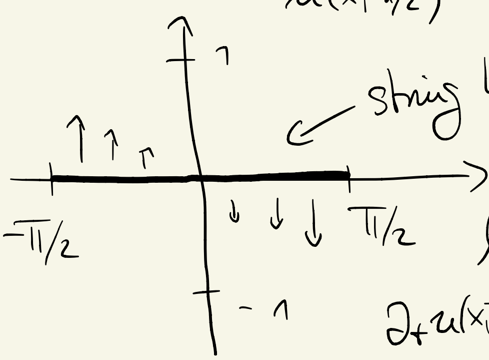
$$(\partial_t u)(x,0) = -\sin(x)\sin(0) = 0$$

$u(x,0)$

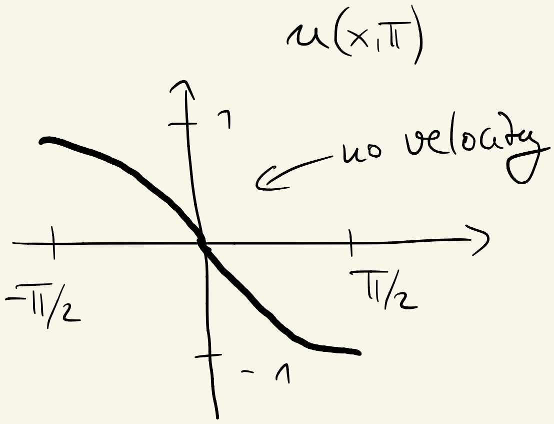


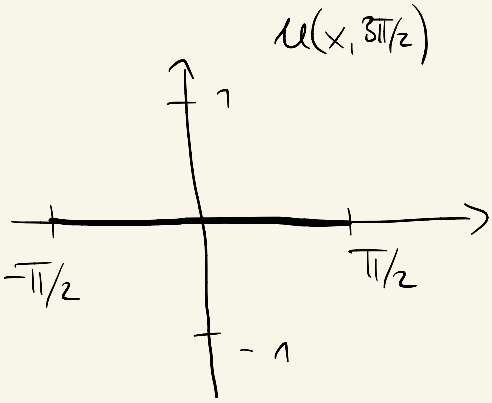
(no initial velocity)

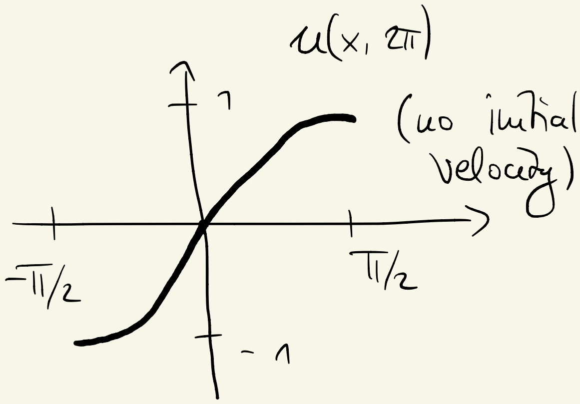
$u(x,\pi/2)$



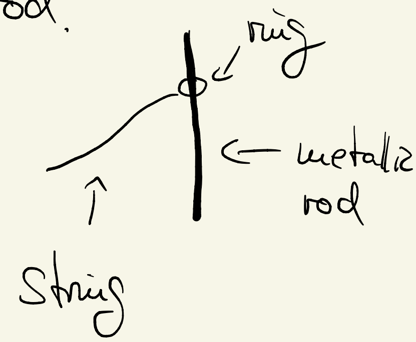
string has a lot of velocity because $\partial_t u(x,\pi/2) = -\sin(x)$







Conclusion: This describes the motion of a string with open ends. Can be constructed by attaching the string to a ring that moves freely up and down a metallic rod.



2.5. Classification of linear second order PDEs

A general linear PDE of second order for a function $u(x,y)$ with $x,y \in \mathbb{R}$ is of the form

$$a(x,y)\, u_{xx} + b(x,y)\, u_{xy} + c(x,y)\, u_{yy}$$
$$+ d(x,y)\, u_x + e(x,y)\, u_y + f(x,y)\, u = g(x,y). \qquad (2.29)$$

Here $a,b,c,d,e,f,g,u : \mathbb{R}^2 \to \mathbb{R}$ and we used the notation

$$u_{xx} = \partial_x^2 u\,, \quad u_x = \partial_x u\,, \quad u_{xy} = \partial_x \partial_y u \qquad (2.30)$$

and so forth. The three second order PDEs we have encountered so far are

(i) $u_t = u_{xx}$, (heat equation)
(ii) $u_{tt} = u_{xx}$, (wave equation)
(iii) $u_{xx} + u_{yy} = 0$, (Laplace equation) (2.31)

or, using the same independent variables, x and y

$$(i) \quad u_{xx} - u_y = 0 \quad \text{(heat eq.)}$$
$$(ii) \quad u_{xx} - u_{yy} = 0 \quad \text{(wave eq.)}$$
$$(iii) \quad u_{xx} + u_{yy} = 0 \quad \text{(Laplace eq.)} \qquad (2.32)$$

Analogous to characterizing quadratic equations

$$ax^2 + bxy + cy^2 + dx + ey + f = 0 \qquad (2.33)$$

as either **hyperbolic**, **parabolic** or **elliptic** determined by

$$b^2 - 4ac > 0, \quad \text{(hyperbolic)}$$
$$b^2 - 4ac = 0, \quad \text{(parabolic)}$$
$$b^2 - 4ac < 0, \quad \text{(elliptic)} \qquad (2.34)$$

we do the same for PDEs.

For the heat equation $a = 1$, $b = 0$, $c = 0$, so $b^2 - 4ac = 0$ and **the heat equation is parabolic**. Similarly, we see that **the wave equation is hyperbolic**, and

Laplace's equation is elliptic.

This leads to a natural question: is it possible to transform (2.29) to another form, where the new PDE is simpler? That is, are there coordinate transformations

$$r(x,y), \quad s(x,y), \tag{2.35}$$

that allows us to transform every linear second order PDE to one of the **standard forms**:

$$\begin{aligned} &u_{rr} - u_{ss} + \text{lots} = 0, && \text{hyperbolic,} \\ &u_{ss} + \text{lots} = 0, && \text{parabolic,} \\ &u_{rr} + u_{ss} + \text{lots} = 0, && \text{elliptic,} \end{aligned} \tag{2.36}$$

where lots stands for **lower order terms**?

Before we provide a general statement, let us discuss the following example.

Example. Let us consider the PDE

$$2u_{xx} - 2u_{xy} + 5u_{yy} = 0. \qquad (2.37)$$

Here $a = 2$, $b = -2$, $c = 5$ and the equation is elliptic because

$$b^2 - 4ac = 4 - 4 \cdot 2 \cdot 5 = -36 < 0. \qquad (2.38)$$

Let us introduce the new variables

$$r(x,y) = 2x + y, \quad s = x - y. \qquad (2.39)$$

An application of the chain rule shows

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} = u_r r_x + u_s s_x,$$

$$u_y = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} = u_r r_y + u_s s_y,$$

$$u_{xx} = u_{rr}(r_x)^2 + u_r r_{xx} + u_{ss}(s_x)^2 + u_s s_{xx} + 2u_{rs} r_x s_x,$$

$$u_{xy} = u_{rr}\, r_x r_y + u_r\, r_{xy} + u_{ss}\, s_x s_y + u_s\, s_{xy} + u_{rs}\left(r_x s_y + r_y s_x\right),$$

$$u_{yy} = u_{rr}(r_y)^2 + u_r\, r_{yy} + u_{ss}(s_y)^2 + u_s\, s_{yy} + 2u_{rs}\, r_y s_y. \qquad (2.40)$$

When we insert the specific form of $r(x,y)$ and $s(x,y)$ from (2.39) into (2.40) this gives

$$
\begin{aligned}
u_{xx} &= 4u_{rr} + 4u_{rs} + u_{ss}, \\
u_{xy} &= 2u_{rr} - u_{rs} - u_{ss}, \\
u_{yy} &= u_{rr} - 2u_{rs} + u_{ss}.
\end{aligned} \qquad (2.41)
$$

Next, we insert (2.41) into (2.37) and find

$$
\begin{aligned}
&2\left(4u_{rr} + 4u_{rs} + u_{ss}\right) - 2\left(2u_{rr} - u_{rs} - u_{ss}\right) \\
&\quad + 5\left(u_{rr} - 2u_{rs} + u_{ss}\right) = 0,
\end{aligned} \qquad (2.42)
$$

which simplifies to

$$u_{rr} + u_{ss} = 0. \qquad (2.43)$$

This is Laplace's equation, which is also elliptic.

General statement: Consider the general linear second order PDE in (2.29) with constant coefficients (a, b, c do <u>not</u> depend on x, y). We have either

$$
\begin{aligned}
&(1) \quad b^2 - 4ac > 0, \quad (\text{parabolic}) \\
\text{or } &(2) \quad b^2 - 4ac = 0, \quad (\text{hyperbolic}) \\
\text{or } &(3) \quad b^2 - 4ac < 0. \quad (\text{elliptic})
\end{aligned}
\tag{2.44}
$$

Then there exists a coordinate transformation $r(x,y)$, $s(x,y)$ such that the equation reads

$$
\begin{aligned}
&(1) \quad u_{rr} - u_{ss} + \text{lots} = 0, \\
&(2) \quad u_{ss} + \text{lots} = 0, \\
&(3) \quad u_{rr} + u_{ss} + \text{lots} = 0,
\end{aligned}
\tag{2.45}
$$

in the new coordinates.

Remark: An alternative form in case (1) is

$$u_{rs} + \text{l.o.t.} = 0. \tag{2.46}$$

To motivate this, we consider

$$u_{tt} - c^2 u_{xx} = 0 \tag{2.47}$$

and introduce the coordinates

$$r(x,t) = x - ct \quad \text{and} \quad s(x,t) = x + ct. \tag{2.48}$$

We insert

$$\begin{aligned} r_x &= 1, \quad r_t = -c, \quad r_{xx} = 0 = r_{tt}, \\ s_x &= 1, \quad s_t = c, \quad s_{xx} = 0 = s_{tt}, \end{aligned} \tag{2.49}$$

into (2.40) and find

$$\begin{aligned} u_{xx} &= u_{rr} + u_{ss} + 2u_{rs}, \\ u_{tt} &= (u_{rr} + u_{ss} - 2u_{rs})\, c^2 \end{aligned} \tag{2.50}$$

When inserted into (2.47) this yields

$$u_{rs} = 0. \tag{2.51}$$

Properties of linear 2nd order PDEs

1. **Elliptic PDEs:** Solutions are as regular as the coefficients allow. E.g. solutions of the Laplace equation $\Delta u(x) = 0$ are analytic where they are defined. Solutions of the Poisson equation $\Delta u(x) = \rho(x)$ are $k+2$ times continuously differentiable if ρ is k times continuously differentiable.

2. **Parabolic PDEs:** Solutions become smooth with time. That is, if $u(x,0)$ is e.g. only continuous but not differentiable then $u(x,t)$ is for all $t>0$ an infinitely differentiable (one also says smooth) function of x and t.

3. **Hyperbolic PDEs:** On page 28 we have seen that the wave equation on $\mathbb{R}$ can describe transport. A consequence is that the solution has exactly

the same regularity as the initial condition. That is, if g (notation of example) is a $\mathcal{C}^k$ function, then the solution is $\mathcal{C}^k$ in space and $\mathcal{C}^k$ in time. In higher dimensions there are solutions with a certain gain in regularity with time. But this is a subtle topic that we cannot discuss here.

General solution to the wave equation on $\mathbb{R}$

On p. 39 we showed that the coordinate transformation (2.48) transforms the wave equation on $\mathbb{R}$ to the form $u_{rs} = 0$. This equation can be integrated:

$$0 = \int_0^b \int_0^a \partial_r \partial_s u(r,s)\, ds\, dr = \int_0^b \left[\partial_s u(a,s) - \partial_s u(0,s)\right] ds$$

$$= u(a,b) - u(a,0) - (\partial_s u)(0,b) + (\partial_s u)(0,0)$$

$$\Leftrightarrow \quad u(a,b) = \underbrace{u(a,0)}_{\substack{\text{function of} \\ \text{first argument} \\ \text{only}}} + \underbrace{(\partial_s u)(0,b)}_{\substack{\text{function of second} \\ \text{argument only}}} - (\partial_s u)(0,0) \qquad (2.52)$$

We conclude that there are two functions $F(r)$ and

$G(s)$ such that

$$u(r,s) = F(r) + G(s), \tag{2.53}$$

whose precise form depends on the initial conditions that we impose. Note however, that without any reference to initial conditions u in (2.53) solves $u_{rs} = 0$. Since $r = x - ct$ and $s = x + ct$ the function

$$u(x,t) = F(x-ct) + G(x+ct) \tag{2.54}$$

solves

$$\partial_t^2 u(x,t) = c^2\, \partial_x^2 u(x,t). \tag{2.55}$$

How do we incorporate an initial condition? Let us assume that $u(x,0) = f(x)$ and $(\partial_t u)(x,0) = g(x)$. Given this information we need to find F and G.

Let's insert:

- $u(x,0) = F(x) + G(x) \overset{!}{=} f(x)$,
- $(\partial_t u)(x,0) = \partial_t \Big(F(x-ct) + G(x+ct) \Big) \Big|_{t=0}$

$$= -c F'(x) + c G'(x) \overset{!}{=} g(x)$$

$$\Leftrightarrow \Big(G'(x) - F'(x) \Big) = \frac{g(x)}{c} \tag{2.56}$$

- Ansatz: $F(x) = \frac{1}{2} f(x) - \frac{1}{2} \eta(x)$

$$G(x) = \frac{1}{2} f(x) + \frac{1}{2} \eta(x) \tag{2.57}$$

This implies $F(x) + G(x) = f(x)$, so the first equation is satisfied. We also have

$$G'(x) - F'(x) = \frac{1}{2}\Big(f'(x) + \eta'(x) - f(x) + \eta'(x) \Big)$$

$$= \eta'(x) \overset{!}{=} \frac{g(x)}{c}$$

$$\Rightarrow \eta(x) = \frac{1}{c} \int_{x_0}^{x} g(y)\, dy \quad \text{for some } x_0 \in \mathbb{R}. \tag{2.58}$$

We conclude that

$$u(x,t) = \frac{1}{2}\Big(f(x-ct) + f(x+ct)\Big) + \frac{1}{2c}\underbrace{\Big(\eta(x+ct) - \eta(x-ct)\Big)}$$

$$= \frac{1}{2c}\left[\int_{x_0}^{x+ct} g(y)\,dy - \int_{x_0}^{x-ct} g(y)\,dy\right]$$

$$= \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy \,. \qquad (2.59)$$

Let us summarize our findings in the following theorem.

Theorem (General solution wave equation on $\mathbb{R}$):

Let $f : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable, let $g : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and consider the initial value problem

$$\begin{cases} \partial_t^2 u(x,t) = c^2 \partial_x^2 u(x,t) & \text{in } \mathbb{R}\times\mathbb{R}_+, \\ u(x,0) = f(x) & \text{on } \mathbb{R}\times\{t=0\}, \\ (\partial_t u)(x,0) = g(x) & \text{on } \mathbb{R}\times\{t=0\}. \end{cases} \tag{2.60}$$

The solution is given by

$$u(x,t) = \frac{1}{2}\Big(f(x-ct) + f(x+ct) \Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy. \tag{2.61}$$

Eq. (2.61) is called **d'Alembert's formula**.

Please insert the initial conditions from our example on p. 28 into (2.61) and check that it gives the correct solution.