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ERROR ANALYSIS OF AN IMMERSED FINITE ELEMENT METHOD FOR EULER-BERNOULLI BEAM INTERFACE PROBLEMS

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Abstract. This article presents an error analysis of a Hermite cubic immersed finite element (IFE) method for solving interface problems of the differential equation modeling a Euler-Bernoulli beam made up of multiple materials together with suitable jump conditions at material interfaces. The analysis consists of three essential groups. The first group is about IFE functions including bounds for the IFE shape functions and inverse inequalities. The second group is about error bounds for IFE interpolation derived with a multi-point Taylor expansion technique. The last group, and perhaps the most important group, is for proving the optimal convergence of the IFE solution generated by the usual Galerkin scheme based on the Hermite cubic IFE space considered in this article.

Key words. Error estimation, interface problem, interface independent mesh, Euler-Bernoulli beam, Hermite cubic finite element, multi-point Taylor expansion, optimal convergence.

1. Introduction

This article presents an error analysis of an immersed finite element (IFE) method that can solve interface problems with interface independent meshes for the differential equation modeling a Euler-Bernoulli beam formed with multiple materials. Without loss of generality, we consider an Euler-Bernoulli beam of length 1 formed with two materials such that the flexural rigidity of this beam is a piecewise positive constant function

$$\beta(x) = \begin{cases} \beta^-, & x \in (0,\alpha), \\ \beta^+, & x \in (\alpha,1), \end{cases} \tag{1}$$

where $\alpha \in \Omega = (0,1)$ is the interface of the two materials whose flexural rigidity are β^- and β^+, respectively. It is well known, see such as [8], that the deflection $u(x)$ of an Euler-Bernoulli beam at a point $x \in \Omega$ corresponding to a given load $f(x)$ satisfies the following so called beam equation:

$$(\beta u''(x))'' = f(x), \ x \in \Omega^- \cup \Omega^+, \tag{2a}$$

where $\Omega^- = (0,\alpha)$ and $\Omega^+ = (\alpha,1)$. Boundary conditions are required to uniquely determine a solution from (2a), and we will consider the clamped boundary conditions at the ends of the beam:

$$u(0) = w_0, \ u'(0) = w_1, \ u(1) = w_3, \ u'(1) = w_4, \tag{2b}$$

other boundary configurations, such as the popular cantilevered boundary conditions can also be considered. At the material interface α, the deflection $u(x)$ is

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required to satisfy the following rigid connection conditions:

$$\text{(2c)} \qquad \begin{cases} u(\alpha-) = u(\alpha+), & \text{(continuity in the deflection)}, \\ u'(\alpha-) = u'(\alpha+), & \text{(continuity in the bending angle)}, \\ \beta^- u''(\alpha-) = \beta^+ u''(\alpha+), & \text{(continuity of the bending moment)}, \\ \beta^- u'''(\alpha-) = \beta^+ u'''(\alpha+), & \text{(continuity of the shear)}. \end{cases}$$

Specifically, the interface problem to be discussed for the Euler-Bernoulli beam equation is to find the deflection function $u(x)$ such that all the equations in (2) are satisfied.

Immersed finite element methods are a group of finite element methods that can solve interface problems with meshes independent of material interfaces where the coefficients in the differential equations are discontinuous. Instead of simple polynomials, IFE methods use Hsieh-Clough-Tocher type macro finite element functions [5, 7] which are piecewise polynomials on each interface element constructed according to interface jump conditions. In other words, IFE methods employ functions that already partially solve interface problems locally on interface elements rather than generic polynomials having nothing to do with a specific interface problem to be solved. Of course, on all the non-interface elements which are not cut by material interfaces, i.e., which are occupied by one of the materials, IFE methods just use polynomials of choice as usual finite element methods. In a certain sense, the fundamental concept for IFE methods has a trace in the generalized finite element method proposed in the 1980s [3, 4] which employs shape functions on an element constructed by locally solving the problem in that element, even though these shape functions may be non-polynomials, they possess key features of the solution to a boundary value problem.

An IFE method was introduced in [13] for solving an interface problem of a two point boundary value problem. Afterwards, IFE methods have been developed for solving elliptic interface problems [2, 10, 12, 14, 16, 19, 25, 26], some time dependent interface problems [11, 22, 18, 20], interface problems for linear elasticity [9, 15, 21, 23], Stokes interface problems [1], and interface problems for some 4-th order differential equations [17]. In particular, a Hermite cubic IFE space has been developed in [27] for solving interface problems of the differential equations modeling a Euler-Bernoulli beam. Fundamental features of this IFE space are presented in [27], such as the unisolvence for the Hermite cubic IFE shape functions on every interface element, the consistence of the IFE shape functions with their corresponding Hermite cubic finite element shape functions. More importantly, it is reported by numerical examples in [27] that this IFE space can be used to produce an optimally convergent approximation to the interface problem of the beam equation. Our goal in this article is to carry out an error analysis for the Hermite cubic IFE space developed in [27] and to theoretically prove that the interpolation in this IFE space has the optimal approximation capability, and the numerical solution produced in this IFE space by the usual Galerkin finite element scheme can converge optimally to the exact solution to the beam interface problem.

In the error analysis to be presented later in this article, we will use the standard Sobolev space on an open subinterval D of Ω: for every integer $m \geq 0$,

$$\text{(3)} \qquad H^m(D) = \{u \mid u^{(j)} \in L^2(D), 0 \leq j \leq m\}$$

on which we have the following norm and semi-norm:

$$\|u\|_{m,D} = \sqrt{\sum_{j=0}^{m} \left\|u^{(j)}\right\|_{0,D}^2}, \ |u|_{m,D} = \left\|u^{(m)}\right\|_{0,D}, \ \forall u \in H^m(D), \tag{4}$$

where $\|\cdot\|_{0,D}$ is the norm of $L^2(D)$. Also, we will use the following related Sobolev space: for every integer $m \geq 1$,

$$H_0^m(D) = \{u \in H^m(D) \mid u^{(j)}|_{\partial D} = 0, \ 0 \leq j \leq m-1\}. \tag{5}$$

In the case when $\alpha \in D$, we let $D^\pm = D \cap \Omega^\pm$ and we will consider the following space:

$$\tilde{H}^m(D) = \{u \mid u|_{D^\pm} \in H^m(D^\pm)\} \tag{6}$$

which is endowed with the following norm and semi-norm:

$$\begin{cases} \|u\|_{m,D} = \sqrt{\sum_{j=0}^{m} \left(\left\|u^{(j)}\right\|_{0,D^-}^2 + \left\|u^{(j)}\right\|_{0,D^+}^2\right)}, \\ |u|_{m,D} = \sqrt{\left(\left\|u^{(m)}\right\|_{0,D^-}^2 + \left\|u^{(m)}\right\|_{0,D^+}^2\right)}, \end{cases} \quad \forall u \in \tilde{H}^m(D). \tag{7}$$

This article is organized as follows. In Section 2, we will recall the Hermite cubic IFE space developed in [27], derive the L^∞ bounds for the Hermite cubic IFE shape functions on every interface element, and establish the inverse inequalities for functions in this IFE space. In Section 3, we use a multi-point Taylor expansion technique to prove that the interpolation in this IFE space has the optimal approximation capability. Section 4 is to show that the IFE solution generated by the usual Galerkin scheme posed on this IFE space converges optimally according to the polynomials used in this IFE space.

2. A Hermite Cubic Immersed Finite Element Space

To be self-contained, let us first recall in this section the Hermite cubic IFE space developed in [17, 27] for solving the interface problem described by (2) for the Euler-Bernoulli beam. Let $\mathcal{T}_h$ be a mesh of the solution domain $\Omega = (0,1)$ with following nodes independent of the interface point α:

$$x_0 = 0 < x_1 < x_2 < \cdots < x_N < x_{N+1} = 1 \tag{8}$$

with

$$h_k = x_{k+1} - x_k, \ k = 0, 1, 2, \ldots, N, \ h = \max_{0 \leq k \leq N} h_k.$$

On each element $e_k = [x_k, x_{k+1}], \ k = 0, 1, \ldots, N$ of the mesh $\mathcal{T}_h$, we have the following standard Hermite cubic finite element shape functions:

$$\phi_{k,j}(x) = \hat{N}_j(F_k(x)), \text{ for } j = 1, 3 \text{ and } \phi_{k,j}(x) = h_k \hat{N}_j(F_k(x)) \text{ for } j = 2, 4, \tag{9}$$

where

$$F_k(x) = \frac{x - x_k}{x_{k+1} - x_k} \tag{10}$$

is the affine mapping between element e_k and the reference element $[0,1]$, and $\hat{N}_j(\xi), j = 1,2,3,4$ are the Hermite cubic shape functions on $[0,1]$ defined by

$$\begin{cases} \hat{N}_1(\xi) = 2\xi^3 - 3\xi^2 + 1, & \hat{N}_2(\xi) = \xi^3 - 2\xi^2 + \xi, \\ \hat{N}_3(\xi) = -2\xi^3 + 3\xi^2, & \hat{N}_4(\xi) = \xi^3 - \xi^2, \end{cases} \text{ for } \xi \in [0,1]. \tag{11}$$

These Hermite cubic finite element shape functions satisfy the following Hermite interpolation conditions:

$$\begin{cases} \phi_{k,1}(x_k) = 1, \phi'_{k,1}(x_k) = 0, \phi_{k,1}(x_{k+1}) = 0, \phi'_{k,1}(x_{k+1}) = 0, \\ \phi_{k,2}(x_k) = 0, \phi'_{k,2}(x_k) = 1, \phi_{k,2}(x_{k+1}) = 0, \phi'_{k,2}(x_{k+1}) = 0, \\ \phi_{k,3}(x_k) = 0, \phi'_{k,3}(x_k) = 0, \phi_{k,3}(x_{k+1}) = 1, \phi'_{k,3}(x_{k+1}) = 0, \\ \phi_{k,4}(x_k) = 0, \phi'_{k,4}(x_k) = 0, \phi_{k,4}(x_{k+1}) = 0, \phi'_{k,4}(x_{k+1}) = 1. \end{cases} \tag{12}$$

When $e_k = [x_k, x_{k+1}] \in \mathcal{T}_h$ is the interface element such that $\alpha \in (x_k, x_{k+1})$, we use α to form two subintervals $\mathring{e}_k^- = (x_k, \alpha)$, $\mathring{e}_k^+ = (\alpha, x_{k+1})$. The following IFE shape functions have been developed in [17, 27]:

$$\begin{cases} \psi_{k,j}(x) = \begin{cases} \psi_{k,j}^-(x) = \widetilde{N}_j^-(F_k(x)), & x \in e_k^-, \\ \psi_{k,j}^+(x) = \widetilde{N}_j^+(F_k(x)), & x \in e_k^+, \end{cases} \text{ for } j = 1,3 \\ \\ \psi_{k,j}(x) = \begin{cases} \psi_{k,j}^-(x) = h_k\widetilde{N}_j^-(F_k(x)), & x \in e_k^-, \\ \psi_{k,j}^+(x) = h_k\widetilde{N}_j^+(F_k(x)), & x \in e_k^+, \end{cases} \text{ for } j = 2,4 \end{cases} \tag{13}$$

where $\widetilde{N}_j(\xi), j = 1,2,3,4$ are the IFE shape functions on the reference element $[0,1]$ such that

$$\begin{aligned} \widetilde{N}_1(\xi) &= \begin{cases} \widetilde{N}_1^-(\xi) = 1 + \xi^2(a_1 + b_1(\xi - \hat{\alpha})) & \text{if } 0 \le \xi \le \hat{\alpha}, \\ \widetilde{N}_1^+(\xi) = (\xi - 1)^2(c_1 + d_1(\xi - \hat{\alpha})) & \text{if } \hat{\alpha} \le \xi \le 1, \end{cases} \\ \widetilde{N}_2(\xi) &= \begin{cases} \widetilde{N}_2^-(\xi) = \xi + \xi^2(a_2 + b_2(\xi - \hat{\alpha})) & \text{if } 0 \le \xi \le \hat{\alpha}, \\ \widetilde{N}_2^+(\xi) = (\xi - 1)^2(c_2 + d_2(\xi - \hat{\alpha})) & \text{if } \hat{\alpha} \le \xi \le 1, \end{cases} \\ \widetilde{N}_3(\xi) &= \begin{cases} \widetilde{N}_3^-(\xi) = \xi^2(a_3 + b_3(\xi - \hat{\alpha})) & \text{if } 0 \le \xi \le \hat{\alpha}, \\ \widetilde{N}_3^+(\xi) = 1 + (\xi - 1)^2(c_3 + d_3(\xi - \hat{\alpha})) & \text{if } \hat{\alpha} \le \xi \le 1, \end{cases} \\ \widetilde{N}_4(\xi) &= \begin{cases} \widetilde{N}_4^-(\xi) = \xi^2(a_4 + b_4(\xi - \hat{\alpha})) & \text{if } 0 \le \xi \le \hat{\alpha}, \\ \widetilde{N}_4^+(\xi) = (\xi - 1) + (\xi - 1)^2(c_4 + d_4(\xi - \hat{\alpha})) & \text{if } \hat{\alpha} \le \xi \le 1, \end{cases} \end{aligned} \tag{14}$$

with $\hat{\alpha} = F_k(\alpha)$ and the coefficients in these shape function are determined by the following linear systems [17, 27]:

$$\begin{bmatrix} \hat{\alpha}^2 & 0 & -(\hat{\alpha}-1)^2 & 0 \\ 2\hat{\alpha} & \hat{\alpha}^2 & -2(\hat{\alpha}-1) & -(\hat{\alpha}-1)^2 \\ 2\beta^- & 4\hat{\alpha}\beta^- & -2\beta^+ & -4(\hat{\alpha}-1)\beta^+ \\ 0 & 6\beta^- & 0 & -6\beta^+ \end{bmatrix} \begin{bmatrix} a_i \\ b_i \\ c_i \\ d_i \end{bmatrix} = \vec{q}_i, \ i = 1,2,3,4, \tag{15}$$

$$\text{with } \vec{q}_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \vec{q}_2 = \begin{bmatrix} -\hat{\alpha} \\ -1 \\ 0 \\ 0 \end{bmatrix}, \ \vec{q}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \vec{q}_4 = \begin{bmatrix} \hat{\alpha} - 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}. \tag{16}$$

The IFE shape functions defined by (13) are macro finite element functions, i.e., they are piecewise cubic polynomials defined according to the subintervals of the interface element $e_k = [x_k, x_{k+1}] \in \mathcal{T}_h$ formed by the interface point α. By design, these IFE shape functions satisfy the standard Hermite interpolation conditions [17, 27]:

$$
\left\{
\begin{array}{l}
\psi_{k,1}(x_k) = 1, \psi'_{k,1}(x_k) = 0, \psi_{k,1}(x_{k+1}) = 0, \psi'_{k,1}(x_{k+1}) = 0, \\
\psi_{k,2}(x_k) = 0, \psi'_{k,2}(x_k) = 1, \psi_{k,2}(x_{k+1}) = 0, \psi'_{k,2}(x_{k+1}) = 0, \\
\psi_{k,3}(x_k) = 0, \psi'_{k,3}(x_k) = 0, \psi_{k,3}(x_{k+1}) = 1, \psi'_{k,3}(x_{k+1}) = 0, \\
\psi_{k,4}(x_k) = 0, \psi'_{k,4}(x_k) = 0, \psi_{k,4}(x_{k+1}) = 0, \psi'_{k,4}(x_{k+1}) = 1.
\end{array}
\right. \tag{17}
$$

Moreover, according to Theorem 2.1 in [17], these IFE shape functions satisfy the jump conditions (2c) in the interface problem, i.e., across the interface $\alpha \in \mathring{e}_k = (x_k, x_{k+1})$, they satisfy

$$
\left[\psi_{k,j}\right]_\alpha = 0,\ \left[\psi'_{k,j}\right]_\alpha = 0,\ \left[\beta\psi''_{k,j}\right]_\alpha = 0,\ \left[\beta\psi'''_{k,}\right]_\alpha = 0,\ j = 1, 2, 3, 4 \tag{18}
$$

where $\left[v\right]_\alpha = v(\alpha+) - v(\alpha-)$ for any function defined on $e_k = [x_k, x_{k+1}]$.

With these Hermite cubic shape functions, we can form the following local IFE space on each element e_k in a mesh $\mathcal{T}_h$:

$$
S_h(e_k) = \begin{cases} span\{\phi_{k,j}(x),\ j = 1, 2, 3, 4\}, & \text{if } e_k \in \mathcal{T}_h \text{ is a non-interface element}, \\ span\{\psi_{k,j}(x),\ j = 1, 2, 3, 4\}, & \text{if } e_k \in \mathcal{T}_h \text{ is an interface element}. \end{cases} \tag{19}
$$

Following the usual procedure, we can use the local IFE spaces defined on elements in a mesh $\mathcal{T}_h$ to form the Hermitian cubic IFE space as follows:

$$
S_h(\Omega) = \{u \in H^2(\Omega) \mid u|_e \in S_h(e),\ \forall e \in \mathcal{T}_h\}. \tag{20}
$$

Even though IFE functions in $S_h(e_k)$ are macro cubic polynomials on the interface element $e_k = [x_k, x_{k+1}] \in \mathcal{T}_h$, they are closely related to their finite element counterparts which are cubic polynomials. For example, the four IFE shape functions $\psi_{k,j}(x),\ j = 1, 2, 3, 4$ are consistent with $\phi_{k,j}(x),\ j = 1, 2, 3, 4$ [17, 27]. Furthermore, some estimates well-known for finite element functions are also valid for IFE functions, among which are L^∞ estimates for the shape functions and inverse inequalities stated in the two lemmas below.

Lemma 2.1. *There exists a constant C independent of the interface location α such that the following estimates hold on the interface element $e_k \in \mathcal{T}_h$:*

$$
\left|\psi_{k,j}^{(m)}(x)\right| \le \begin{cases} Ch_k^{-m}, & j = 1, 3, \\ Ch_k^{1-m}, & j = 2, 4, \end{cases} \quad m = 0, 1, 2, 3,\ \forall x \in e_k. \tag{21}
$$

Proof. We use the formulas in (15) and (16) to compute a_j, b_j, c_j and d_j for $j = 1, 2, 3, 4$ in (14). Without loss of generality, we assume that $e_k = [0, h_k]$ such that $\alpha = ah_k$ and $x = bh_k$ with $a, b \in [0, 1]$, and we let $e_k^- = [0, \alpha]$ and $e_k^+ = [\alpha, h_k]$. With these preparations, using direct algebraic calculations and by (13), we can show that

(22)
$$\left|\psi_{k,j}^{(m)}(x)\right| = \begin{cases} \dfrac{\left|N_{j,m}^{s}(a,b,\beta^-,\beta^+)\right|}{D(a,\beta^-,\beta^+)} h_k^{-m}, & x \in e_k^s,\ s=-,+,\ j=1,3, \\[2ex] \dfrac{\left|N_{j,m}^{s}(a,b,\beta^-,\beta^+)\right|}{D(a,\beta^-,\beta^+)} h_k^{1-m}, & x \in e_k^s,\ s=-,+,\ j=2,4, \end{cases} \qquad m=0,1,2,3$$

where $N_{j,m}^s(a,b,\beta^-,\beta^+), s=-,+$ and $D(a,\beta^-,\beta^+)$ are polynomials in terms of a, b, β^- and β^+. In particular,

$$D(a,\beta^-,\beta^+) \;=\; (-1+a)^4(\beta^-)^2 - 2(-1+a)a(2+(-1+a)a)\beta^-\beta^+ + a^4(\beta^+)^2.$$

Thus, by the fact that $a \in [0,1]$, we have

$$D(a,\beta^-,\beta^+) \geq (-1+a)^4(\beta^-)^2 + a^4(\beta^+)^2 \geq C(\beta^-,\beta^+) > 0,\ \forall a \in [0,1]. \tag{23}$$

Hence, (21) follows from applying (23) to (22).

■

Lemma 2.2. *There exists a constant C independent of α such that the following inverse inequalities hold on the interface element $e_k \in \mathcal{T}_h$ for nonnegative integers l and m satisfying $0 \leq l \leq m \leq 3$:*

$$|v|_{m,\mathring{e}_k} \leq C h_k^{l-m} |v|_{l,\mathring{e}_k},\ \forall\, v \in S_h(\mathring{e}_k), \tag{24}$$

here, the semi-norm is defined in (7).

Proof. Estimates given in (24) are obvious when $l = m$. Then, we consider the case in which $l = m-1, m = 1,2,3$. Without loss of generality, we assume the interface element is $e_k = [0, h_k]$, let $e_k^- = [0,\alpha]$ and $e_k^+ = [\alpha, h_k]$, and assume that $\tilde{e}_k = [0, h_k/2] \subseteq e_k^-$. Every IFE function $v \in S_h(\mathring{e}_k)$ can be expressed as

$$v(x) = \begin{cases} v^-(x) = c_0^- + c_1^- x + c_2^- x^2 + c_3^- x^3,\ x \in e_k^-, \\ v^+(x) = c_0^+ + c_1^+ x + c_2^+ x^2 + c_3^+ x^3,\ x \in e_k^+. \end{cases}$$

The jump conditions in (18) implies that coefficients of v satisfy $\mathbf{M}^-\mathbf{C}^- = \mathbf{M}^+\mathbf{C}^+$ with

$$\mathbf{M}^- = \begin{pmatrix} 1 & \alpha & \alpha^2 & \alpha^3 \\ 0 & 1 & 2\alpha & 3\alpha^2 \\ 0 & 0 & 2\beta^- & 6\beta^-\alpha \\ 0 & 0 & 0 & 6\beta^- \end{pmatrix},\ \mathbf{M}^+ = \begin{pmatrix} 1 & \alpha & \alpha^2 & \alpha^3 \\ 0 & 1 & 2\alpha & 3\alpha^2 \\ 0 & 0 & 2\beta^+ & 6\beta^+\alpha \\ 0 & 0 & 0 & 6\beta^+ \end{pmatrix},\ \mathbf{C}^\pm = \begin{pmatrix} c_0^\pm \\ c_1^\pm \\ c_2^\pm \\ c_3^\pm \end{pmatrix}.$$

Noted that upper triangular matrices $\mathbf{M}^\pm$ are invertible so that

$$\mathbf{C}^- = (\mathbf{M}^-)^{-1}\mathbf{M}^+\mathbf{C}^+ \text{ and } \mathbf{C}^+ = (\mathbf{M}^+)^{-1}\mathbf{M}^-\mathbf{C}^-.$$

In addition, we have $(\mathbf{M}^-)^{-1}\mathbf{M}^+$ and $(\mathbf{M}^+)^{-1}\mathbf{M}^-$ are both upper triangular matrices, and each entry of the matrices can be bounded above by a constant C independent of α because $0 \leq \alpha \leq h_k \leq 1$. Therefore, we have

$$\begin{aligned} &|c_0^\pm| \leq C(|c_0^\mp| + \alpha^2|c_2^\mp| + \alpha^3|c_3^\mp|), \\ &|c_1^\pm| \leq C(|c_1^\mp| + \alpha|c_2^\mp| + \alpha^2|c_3^\mp|), \\ &|c_2^\pm| \leq C|c_2^\mp|, \\ &|c_3^\pm| \leq C|c_3^\mp|. \end{aligned} \tag{25}$$

Using the above inequalities, we have

$$
\begin{aligned}
|v|^2_{1,\mathring{e}_k} &= \int_0^\alpha (c_1^- + 2c_2^- x + 3c_3^- x^2)^2\,dx + \int_\alpha^{h_k} (c_1^+ + 2c_2^+ x + 3c_3^+ x^2)^2\,dx \\
&\le h_k(\left|c_1^-\right| + 2\left|c_2^-\right| h_k + 3\left|c_3^-\right| h_k^2)^2 + h_k(\left|c_1^+\right| + 2\left|c_2^+\right| h_k + 3\left|c_3^+\right| h_k^2)^2 \\
&\le Ch_k\left((c_1^-)^2 + (c_2^-)^2 h_k^2 + (c_3^-)^2 h_k^4\right).
\end{aligned} \tag{26}
$$

By direct calculations, we have

$$
\begin{aligned}
|v|^2_{1,\mathring{\tilde{e}}_k} &= \int_0^{\frac{h_k}{2}} (c_1^- + 2c_2^- x + 3c_3^- x^2)^2\,dx \\
&= \frac{h_k}{2}\left((c_1^-)^2 + \frac{1}{3}(c_2^-)^2 h_k^2 + \frac{9}{80}(c_3^-)^2 h_k^4 + c_1^- c_2^- h_k + \frac{1}{2} c_1^- c_3^- h_k^2 + \frac{3}{8} c_2^- c_3^- h_k^3\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&1024\,|v|^2_{1,\mathring{\tilde{e}}_k} - h_k\left((c_1^-)^2 + (c_2^-)^2 h_k^2 + (c_3^-)^2 h_k^4\right) \\
=&511 h_k (c_1^-)^2 + \frac{509}{3} h_k^3 (c_2^-)^2 + \frac{283}{5} h_k^5 (c_3^-)^2 \\
&+ 512 h_k^2 c_1^- c_2^- + 256 h_k^3 c_1^- c_3^- + 192 h_k^4 c_2^- c_3^-.
\end{aligned} \tag{27}
$$

Note that the right hand side of (27) is a quadratic form about c_j^-, $j = 1, 2, 3$ whose associated matrix is

$$
M = \begin{bmatrix} 511 h_k & 256 h_k^2 & 128 h_k^3 \\ 256 h_k^2 & (509/3) h_k^3 & 96 h_k^4 \\ 128 h_k^3 & 96 h_k^4 & (283/5) h_k^5 \end{bmatrix},
$$

and it is easy to verify that M is a symmetric positive definite matrix. Therefore, (27) leads to

$$
h_k\left((c_1^-)^2 + (c_2^-)^2 h_k^2 + (c_3^-)^2 h_k^4\right) \le 1024\,|v|^2_{1,\mathring{\tilde{e}}_k}.
$$

Applying this inequality to (26), we have

$$
|v|^2_{1,\mathring{e}_k} \le C\,|v|^2_{1,\mathring{\tilde{e}}_k}. \tag{28}
$$

By the direct calculation and (25), we obtain

$$
|v|^2_{2,\mathring{\tilde{e}}_k} = 2(c_2^-)^2 h_k + \frac{3}{2}(c_3^-)^2 h_k^3 + 3c_2^- c_3^- h_k^2 \ge (2 - \frac{3}{4\sigma_1})(c_2^-)^2 h_k + (\frac{3}{2} - 3\sigma_1)(c_3^-)^2 h_k^3,
$$

for any positive constant σ_1, and

$$
|v|^2_{2,\mathring{e}_k} = \int_0^\alpha (2c_2^- + 6c_3^- x)^2\,dx + \int_\alpha^{h_k} (2c_2^+ + 6c_3^+ x)^2\,dx \le C\left((c_2^-)^2 h_k + (c_3^-)^2 h_k^3\right).
$$

Using a $\sigma_1 \in (\frac{3}{8}, \frac{1}{2})$, we have

$$
|v|^2_{2,\mathring{e}_k} \le C\left((c_2^-)^2 h_k + (c_3^-)^2 h_k^3\right) \le C\,|v|^2_{2,\mathring{\tilde{e}}_k}. \tag{29}
$$

Similarly, we have

$$
\begin{aligned}
|v|^2_{3,\mathring{e}_k} &= \int_0^\alpha (6c_3^-)^2\,dx + \int_\alpha^{h_k} (6c_3^+)^2\,dx \le 36((c_3^-)^2 + (c_3^+)^2) h_k \le C(c_3^-)^2 h_k, \\
|v|^2_{3,\mathring{\tilde{e}}_k} &= \int_0^{\frac{h_k}{2}} (6c_3^-)^2\,dx = 18(c_3^-)^2 h_k,
\end{aligned}
$$

which implies

$$
|v|^2_{3,\mathring{e}_k} \le C\,|v|^2_{3,\mathring{\tilde{e}}_k}. \tag{30}
$$

Finally, estimates (28), (29) and (30) together with the standard inverse inequalities on $\tilde{e}_k$ lead to

$$|v|_{m,\mathring{e}_k} \le C\,|v|_{m,\mathring{\tilde{e}}_k} \le Ch_k^{-1}\,|v|_{m-1,\mathring{\tilde{e}}_k} \le Ch_k^{-1}\,|v|_{m-1,\mathring{e}_k}\,,\ m=1,2,3, \tag{31}$$

and (24) follows from applying (31) repeatedly.

■

3. The Approximation Capability of the Hermite Cubic IFE Space

It has been demonstrated by numerical examples in [27] that the Hermite cubic IFE space $S_h(\Omega)$ has the optimal approximation capability, i.e., the interpolation of a function $u(x)$ in the IFE space $S_h(\Omega)$ can approximate $u(x)$ optimally provided that the function $u(x)$ has a suitable regularity. Our goal in this section is to theoretically confirm this optimal approximation capability by an error estimation for interpolations in this IFE space.

Let $u(x)$ be a function in $H^2(\Omega)$. First, locally on an element $e_k = [x_k, x_{k+1}]$ in the mesh $\mathcal{T}_h$ used to construct the IFE space $S_h(\Omega)$, if e_k is a non-interface element, we define the local IFE interpolation of $u(x)$ as the following cubic polynomial:

$$\begin{aligned} I_{h,k}u(x) =& u(x_k)\phi_{k,1}(x) + u'(x_k)\phi_{k,2}(x) \\ &+ u(x_{k+1})\phi_{k,3}(x) + u'(x_{k+1})\phi_{k,4}(x),\ \forall x \in e_k, \end{aligned} \tag{32}$$

but if e_k is an interface element, we define the local IFE interpolation of $u(x)$ as the following macro/piecewise cubic polynomial:

$$\begin{aligned} I_{h,k}u(x) =& u(x_k)\psi_{k,1}(x) + u'(x_k)\psi_{k,2}(x) \\ &+ u(x_{k+1})\psi_{k,3}(x) + u'(x_{k+1})\psi_{k,4}(x),\ \forall x \in e_k. \end{aligned} \tag{33}$$

Then, globally over $\overline{\Omega} = [0,1]$, we define the IFE interpolation of $u(x)$ to be the piecewise cubic polynomial $I_h u(x) \in S_h(x)$ such that

$$I_h u(x) = I_{h,k}u(x),\ \forall x \in e_k \in \mathcal{T}_h. \tag{34}$$

We start from the following estimates for the error in $I_h u(x)$ on a non-interface element $e_k \in \mathcal{T}_h$:

$$\left\|(I_h u)^{(m)} - u^{(m)}\right\|_{0,\mathring{e}_k} \le Ch_k^{4-m}\,|u|_{4,\mathring{e}_k}\,,\ m=0,1,2,3, \tag{35}$$

which requires $u|_{\mathring{e}_k} \in H^4(\mathring{e}_k)$, where $\mathring{e}_k = (x_k, x_{k+1})$. These error estimates follow from a standard textbook numerical analysis, see [24] for example. However, these estimates are not valid on the interface element for any function $u(x)$ satisfying the jump conditions given in (2c) because u is not a H^4 function in general unless $\beta^- = \beta^+$. Instead, we will derive optimal error estimates for $I_h u$ on the interface element by extending the multi-point Taylor expansion idea [6] to piecewise H^4 functions, and we proceed as follows.

Hereinafter in this section, we let $e_k = [x_k, x_{k+1}] \in \mathcal{T}_h$ be the interface element such that interface point $\alpha \in (x_k, x_{k+1})$. The following lemma provides formulas to represent values of u and u' at the boundary points of e_k in terms of u and its derivatives at $x \in e_k^- = [x_k, \alpha]$.

Lemma 3.1. *Let u be a function in $\tilde{H}^4(\mathring{e}_k)$ and assume it satisfies the jump conditions in* (2c). *Then, for $x \in e_k^-$ we have*

$$u(x_k) = C_1^- + R_1^-, \tag{36a}$$

$$C_1^- = u(x) + u'(x)(x_k - x) + \frac{1}{2}u''(x)(x_k - x)^2 + \frac{1}{6}u'''(x)(x_k - x)^3, \tag{36b}$$

$$R_1^- = \frac{1}{6}\int_x^{x_k} (x_k - s)^3 u^{(4)}(s)ds, \tag{36c}$$

$$u'(x_k) = C_2^- + R_2^-, \tag{36d}$$

$$C_2^- = u'(x) + u''(x)(x_k - x) + \frac{1}{2}u'''(x)(x_k - x)^2, \tag{36e}$$

$$R_2^- = \frac{1}{2}\int_x^{x_k} (x_k - s)^2 u^{(4)}(s)ds, \tag{36f}$$

$$u(x_{k+1}) = C_3^- + R_3^-, \tag{36g}$$

$$\begin{aligned} C_3^- &= u(x) + u'(x)(x_{k+1} - x) + \frac{1}{2}u''(x)(x_{k+1} - x)^2 + \frac{1}{6}u'''(x)(x_{k+1} - x)^3 \\ &\quad + \frac{1}{2}\left(-1 + \frac{\beta^-}{\beta^+}\right)(u''(x) + u'''(x)(\alpha - x))(x_{k+1} - \alpha)^2 \\ &\quad + \frac{1}{6}\left(-1 + \frac{\beta^-}{\beta^+}\right)u'''(x)(x_{k+1} - \alpha)^3, \end{aligned} \tag{36h}$$

$$\begin{aligned} R_3^- &= R_{3,1}^- + R_{3,2}^- + R_{3,3}^-, \\ R_{3,1}^- &= \frac{1}{6}\int_x^{x_{k+1}} (x_{k+1} - s)^3 u^{(4)}(s)ds, \\ R_{3,2}^- &= \frac{1}{2}\left(-1 + \frac{\beta^-}{\beta^+}\right)(x_{k+1} - \alpha)^2 \int_x^{\alpha} (\alpha - s)u^{(4)}(s)ds, \\ R_{3,3}^- &= \frac{1}{6}\left(-1 + \frac{\beta^-}{\beta^+}\right)(x_{k+1} - \alpha)^3 \int_x^{\alpha} u^{(4)}(s)ds, \end{aligned} \tag{36i}$$

$$u'(x_{k+1}) = C_4^- + R_4^-, \tag{36j}$$

$$\begin{aligned} C_4^- &= u'(x) + u''(x)(x_{k+1} - x) + \frac{1}{2}u'''(x)(x_{k+1} - x)^2 \\ &\quad + \left(-1 + \frac{\beta^-}{\beta^+}\right)(u''(x) + u'''(x)(\alpha - x))(x_{k+1} - \alpha) \\ &\quad + \frac{1}{2}\left(-1 + \frac{\beta^-}{\beta^+}\right)u'''(x)(x_{k+1} - \alpha)^2, \end{aligned} \tag{36k}$$

$$\begin{aligned} R_4^- &= R_{4,1}^- + R_{4,2}^- + R_{4,3}^-, \\ R_{4,1}^- &= \frac{1}{2}\int_x^{x_{k+1}} (x_{k+1} - s)^2 u^{(4)}(s)ds, \\ R_{4,2}^- &= \left(-1 + \frac{\beta^-}{\beta^+}\right)(x_{k+1} - \alpha)\int_x^{\alpha} (\alpha - s)u^{(4)}(s)ds, \\ R_{4,3}^- &= \frac{1}{2}\left(-1 + \frac{\beta^-}{\beta^+}\right)(x_{k+1} - \alpha)^2 \int_x^{\alpha} u^{(4)}(s)ds. \end{aligned} \tag{36l}$$

Proof. Expansions (36a) and (36d) are standard Taylor expansions with integral remainders. For (36g), since $u \in \tilde{H}^4(\mathring{e}_k)$ and u satisfies the jump conditions given in (2c), we know that $u \in C^1(e_k)$ and

$$\begin{aligned}
&u(x_{k+1}) = u(x) + \int_x^{x_{k+1}} u'(s)ds \\
&= u(x) - \int_x^{\alpha} u'(s)d(x_{k+1}-s) - \int_\alpha^{x_{k+1}} u'(s)d(x_{k+1}-s) \\
&= u(x) - \left(u'(s)(x_{k+1}-s)|_x^\alpha - \int_x^\alpha (x_{k+1}-s)u''(s)ds \right) \\
&\quad - \left(u'(s)(x_{k+1}-s)|_\alpha^{x_{k+1}} - \int_\alpha^{x_{k+1}} (x_{k+1}-s)u''(s)ds \right) \\
&= u(x) + u'(x)(x_{k+1}-x) - \frac{1}{2}\int_x^\alpha u''(s)d(x_{k+1}-s)^2 - \frac{1}{2}\int_\alpha^{x_{k+1}} u''(s)d(x_{k+1}-s)^2.
\end{aligned}$$

Then, applying integration by parts and the jump condition about u'' in (2c), we have

$$\begin{aligned}
u(x_{k+1}) &= u(x) + u'(x)(x_{k+1}-x) + \frac{1}{2}u''(x)(x_{k+1}-x)^2 \\
&\quad + \frac{1}{2}\left(-1 + \frac{\beta^-}{\beta^+}\right) u''(\alpha-)(x_{k+1}-\alpha)^2 \\
&\quad - \frac{1}{6}\int_x^\alpha u'''(s)d(x_{k+1}-s)^3 - \frac{1}{6}\int_\alpha^{x_{k+1}} u'''(s)d(x_{k+1}-s)^3.
\end{aligned}$$

Applying integration by parts one more time and the jump condition about u''' in (2c), we have

$$\begin{aligned}
u(x_{k+1}) &= u(x) + u'(x)(x_{k+1}-x) + \frac{1}{2}u''(x)(x_{k+1}-x)^2 \\
&\quad + \frac{1}{6}u'''(x)(x_{k+1}-x)^3 + \frac{1}{2}\left(-1 + \frac{\beta^-}{\beta^+}\right) u''(\alpha-)(x_{k+1}-\alpha)^2 \\
&\quad + \frac{1}{6}\left(-1 + \frac{\beta^-}{\beta^+}\right) u'''(\alpha-)(x_{k+1}-\alpha)^3 + \frac{1}{6}\int_x^{x_{k+1}} (x_{k+1}-s)^3 u^{(4)}(s)ds. \quad (37)
\end{aligned}$$

Finally, we obtain (36g) by applying the following standard Taylor expansions to (37):

$$\begin{aligned}
u''(\alpha-) &= u''(x) + u'''(x)(\alpha - x) + \int_x^\alpha (\alpha - s)u^{(4)}(s)ds, \\
u'''(\alpha-) &= u'''(x) + \int_x^\alpha u^{(4)}(s)ds.
\end{aligned}$$

Expansion (36j) can be derived by similar procedures.

■

Similarly, the following lemma provides formulas to represent values of u and u' at the boundary points of e_k in terms of u and its derivatives at $x \in e_k^+ = [\alpha, x_{k+1}]$.

Lemma 3.2. *Let u be a function in $\tilde{H}^4(\mathring{e}_k)$ and assume it satisfies the jump conditions in* (2c). *Then, for $x \in e_k^+$ we have*

$$u(x_k) = C_1^+ + R_1^+, \tag{38a}$$

$$\begin{aligned} C_1^+ &= u(x) + u'(x)(x_k - x) + \frac{1}{2}u''(x)(x_k - x)^2 + \frac{1}{6}u'''(x)(x_k - x)^3 \\ &\quad + \frac{1}{2}\left(-1 + \frac{\beta^+}{\beta^-}\right)(u''(x) + u'''(x)(\alpha - x))(x_k - \alpha)^2 \\ &\quad + \frac{1}{6}\left(-1 + \frac{\beta^+}{\beta^-}\right)u'''(x)(x_k - \alpha)^3, \end{aligned} \tag{38b}$$

$$R_1^+ = R_{1,1}^+ + R_{1,2}^+ + R_{1,3}^+, \tag{38c}$$

$$\begin{aligned} R_{1,1}^+ &= \frac{1}{6}\int_x^{x_k}(x_k - s)^3 u^{(4)}(s)ds, \\ R_{1,2}^+ &= \frac{1}{2}\left(-1 + \frac{\beta^+}{\beta^-}\right)(x_k - \alpha)^2 \int_x^{\alpha}(\alpha - s)u^{(4)}(s)ds, \\ R_{1,3}^+ &= \frac{1}{6}\left(-1 + \frac{\beta^+}{\beta^-}\right)(x_k - \alpha)^3 \int_x^{\alpha} u^{(4)}(s)ds, \end{aligned}$$

$$u'(x_k) = C_2^+ + R_2^+, \tag{38d}$$

$$\begin{aligned} C_2^+ &= u'(x) + u''(x)(x_k - x) + \frac{1}{2}u'''(x)(x_k - x)^2 \\ &\quad + \left(-1 + \frac{\beta^+}{\beta^-}\right)(u''(x) + u'''(x)(\alpha - x))(x_k - \alpha) \\ &\quad + \frac{1}{2}\left(-1 + \frac{\beta^+}{\beta^-}\right)u'''(x)(x_k - \alpha)^2, \end{aligned} \tag{38e}$$

$$R_2^+ = R_{2,1}^+ + R_{2,2}^+ + R_{2,3}^+, \tag{38f}$$

$$\begin{aligned} R_{2,1}^+ &= \frac{1}{2}\int_x^{x_k}(x_k - s)^2 u^{(4)}(s)ds, \\ R_{2,2}^+ &= \left(-1 + \frac{\beta^+}{\beta^-}\right)(x_k - \alpha)\int_x^{\alpha}(\alpha - s)u^{(4)}(s)ds, \\ R_{2,3}^+ &= \frac{1}{2}\left(-1 + \frac{\beta^+}{\beta^-}\right)(x_k - \alpha)^2 \int_x^{\alpha} u^{(4)}(s)ds. \end{aligned}$$

$$u(x_{k+1}) = C_3^+ + R_3^+, \tag{38g}$$

$$C_3^+ = u(x) + u'(x)(x_{k+1} - x) + \frac{1}{2}u''(x)(x_{k+1} - x)^2 + \frac{1}{6}u'''(x)(x_{k+1} - x)^3, \tag{38h}$$

$$R_3^+ = \frac{1}{6}\int_x^{x_{k+1}}(x_{k+1} - s)^3 u^{(4)}(s)ds, \tag{38i}$$

$$u'(x_{k+1}) = C_4^+ + R_4^+, \tag{38j}$$

$$C_4^+ = u'(x) + u''(x)(x_{k+1} - x) + \frac{1}{2}u'''(x)(x_{k+1} - x)^2, \tag{38k}$$

$$R_4^+ = \frac{1}{2}\int_x^{x_{k+1}}(x_{k+1} - s)^2 u^{(4)}(s)ds. \tag{38l}$$

Proof. Expansions (38a), (38d), (38g) and (38j) follow from similar arguments used in the proof for Lemma 3.1.

■

On the interface element e_k, putting expansions derived in Lemmas 3.1 and 3.2 into the IFE interpolation defined by (33), we have

$$
\begin{aligned}
(I_{h,k}u)^{(m)}(x) &= u(x_k)\psi_{k,1}^{(m)}(x) + u'(x_k)\psi_{k,2}^{(m)}(x) + u(x_{k+1})\psi_{k,3}^{(m)}(x) + u'(x_{k+1})\psi_{k,4}^{(m)}(x) \\
&= \sum_{j=1}^{4} C_j^s \frac{d^m\psi_{k,j}^s(x)}{dx^m} + \sum_{j=1}^{4} R_j^s \frac{d^m\psi_{k,j}^s(x)}{dx^m},\ m = 0,1,2,3,\ x \in e_k^s,\ s = -,+. \qquad (39)
\end{aligned}
$$

The following lemma shows that how the term involving $C_j^s, 1 \le j \le 4, s = -,+$ in (39) can be simplified.

Lemma 3.3. *Let u be a function that satisfies the conditions stated in Lemmas 3.1 and 3.2. Then*

$$
\sum_{j=1}^{4} C_j^s \frac{d^m\psi_{k,j}^s(x)}{dx^m} = u^{(m)}(x),\ m = 0,1,2,3,\ x \in e_k^s,\ s = -,+. \qquad (40)
$$

Proof. We give a proof for the case in which $m = 0$ and $x \in e_k^-$, similar argument can be applied to other cases. First we note that $C_j^-, j = 1,2,3,4$ are linear functionals of $u(x), u'(x), u''(x)$ and $u'''(x)$, and by elementary algebraic calculations, we can rewrite $\sum_{j=1}^4 C_j^- \psi_{k,j}^-(x)$ as follows:

(41)

$$
\sum_{j=1}^{4} C_j^- \psi_{k,j}^-(x) = \quad D_1(x)u(x) + D_2(x)u'(x) + D_3(x)u''(x) + D_4(x)u'''(x),\ x \in e_k^-
$$

with

$$
\begin{aligned}
D_1(x) &= \psi_{k,1}^-(x) + \psi_{k,3}^-(x), \\
D_2(x) &= (x_k - x)\psi_{k,1}^-(x) + \psi_{k,2}^-(x) + (x_{k+1} - x)\psi_{k,3}^-(x) + \psi_{k,4}^-(x), \\
D_3(x) &= \frac{1}{2}(x_k - x)^2\psi_{k,1}^-(x) + (x_k - x)\psi_{k,2}^-(x) \\
&\quad + \left[\frac{1}{2}(x_{k+1} - x)^2 + \frac{1}{2}\left(-1 + \frac{\beta^-}{\beta^+}\right)(x_{k+1} - \alpha)^2\right]\psi_{k,3}^-(x) \\
&\quad + \left[(x_{k+1} - x) + \left(-1 + \frac{\beta^-}{\beta^+}\right)(x_{k+1} - \alpha)\right]\psi_{k,4}^-(x), \\
D_4(x) &= \frac{1}{6}(x_k - x)^3\psi_{k,1}^-(x) + \frac{1}{2}(x_k - x)^2\psi_{k,2}^-(x) \\
&\quad + \left[\frac{1}{6}(x_{k+1} - x)^3 + \frac{1}{2}\left(-1 + \frac{\beta^-}{\beta^+}\right)(\alpha - x)(x_{k+1} - \alpha)^2\right. \\
&\qquad\qquad \left. + \frac{1}{6}\left(-1 + \frac{\beta^-}{\beta^+}\right)(x_{k+1} - \alpha)^3\right]\psi_{k,3}^-(x) \\
&\quad + \left[\frac{1}{2}(x_{k+1} - x)^2 + \left(-1 + \frac{\beta^-}{\beta^+}\right)(\alpha - x)(x_{k+1} - \alpha)\right. \\
&\qquad\qquad \left. + \frac{1}{2}\left(-1 + \frac{\beta^-}{\beta^+}\right)(x_{k+1} - \alpha)^2\right]\psi_{k,4}^-(x).
\end{aligned}
$$

We can easily see that $D_j(x), j = 1,2,3,4$ are polynomials whose degree are not more than 6. By the fact that $1 \in S_h(e_k)$, we have

$$
D_1(x) = \psi_{k,1}^-(x) + \psi_{k,3}^-(x) = 1.
$$

In addition, by the fact that $1, x \in S_h(e_k)$, we have

$$\begin{aligned} D_2(x) &= (x_k - x)\psi_{k,1}^-(x) + \psi_{k,2}^-(x) + (x_{k+1} - x)\psi_{k,3}^-(x) + \psi_{k,4}^-(x) \\ &= x_k\psi_{k,1}^-(x) + \psi_{k,2}^-(x) + x_{k+1}\psi_{k,3}^-(x) + \psi_{k,4}^-(x) - x(\psi_{k,1}^-(x) + \psi_{k,3}^-(x)) \\ &= x - x = 0. \end{aligned}$$

Furthermore, by using formulas of the IFE shape functions given in (13) and direct algebraic calculations, we can verify that $D_3(x) = 0$ and $D_4(x) = 0$. Finally, putting these values of $D_j(x), j = 1, 2, 3, 4$ in (41), we obtain (40) for the case in which $m = 0$ and $x \in [x_k, \alpha]$.

∎

By the three lemmas above and (39), we have the following expansion for the IFE interpolation error on the interface element e_k:

(42)
$$(I_{h,k}u)^{(m)}(x) - u^{(m)}(x) = \begin{cases} \sum_{j=1}^{4} R_j^- \dfrac{d^m\psi_{k,j}^-(x)}{dx^m}, & x \in e_k^- = [x_k, \alpha], \\ \sum_{j=1}^{4} R_j^+ \dfrac{d^m\psi_{k,j}^+(x)}{dx^m}, & x \in e_k^+ = [\alpha, x_{k+1}]. \end{cases} \quad m = 0, 1, 2, 3.$$

This expansion suggests us to estimate the magnitudes of the four terms on the right hand of (42) in order to derive an estimate for the error of the IFE interpolation, and we proceed in the following theorem.

Theorem 3.1. *There exists a constant C independent of the interface location α such that for every $u \in \tilde{H}^4(\mathring{e}_k)$ satisfying the jump conditions given in* (2c), *where $e_k \in \mathcal{T}_h$ be the interface element, we have the following estimates for the IFE interpolation defined by* (33)*:*

$$\left\|(I_{h,k}u)^{(m)} - u^{(m)}\right\|_{0,\mathring{e}_k} \le Ch_k^{4-m}\big(|u|_{4,\mathring{e}_k^-} + |u|_{4,\mathring{e}_k^+}\big),\ m = 0, 1, 2, 3. \tag{43}$$

Proof. We start from the estimation on e_k^-. By Lemma 2.1 and the formula for R_1^- given in Lemma 3.1, we have

$$\begin{aligned} \left\|R_1^- \frac{d^m\psi_{k,1}^-}{dx^m}\right\|_{0,\mathring{e}_k^-} &= \left(\int_{x_k}^{\alpha}\left(\frac{1}{6}\int_x^{x_k}(x_k - s)^3u^{(4)}(s)ds\frac{d^m\psi_{k,1}^-(x)}{dx^m}\right)^2 dx\right)^{1/2} \\ &\le Ch_k^{-m}\left(\int_{x_k}^{\alpha}\left(\int_{x_k}^{x}(x_k - s)^3u^{(4)}(s)ds\right)^2 dx\right)^{1/2} \\ &\le Ch_k^{-m}\left(\int_{x_k}^{\alpha}\left(\int_{x_k}^{x}(x_k - s)^6ds\int_{x_k}^{x}\left|u^{(4)}(s)\right|^2 ds\right) dx\right)^{1/2} \\ &\le Ch_k^{-m}\left(\int_{x_k}^{\alpha}\left(\int_{x_k}^{\alpha}(x_k - s)^6ds\int_{x_k}^{\alpha}\left|u^{(4)}(s)\right|^2 ds\right) dx\right)^{1/2} \\ &\le Ch_k^{4-m}\,|u|_{4,\mathring{e}_k^-}. \end{aligned}$$

Similarly, by Lemma 2.1 and the formula for R_2^- given in Lemma 3.1, we have

$$\begin{aligned}
\left\| R_2^- \frac{d^m \psi_{k,2}^-}{dx^m} \right\|_{0,\mathring{e}_k^-} &= \left(\int_{x_k}^{\alpha} \left(\frac{1}{2} \int_x^{x_k} (x_k - s)^2 u^{(4)}(s) ds \frac{d^m \psi_{k,2}^-(x)}{dx^m} \right)^2 dx \right)^{1/2} \\
&\le C h_k^{1-m} \left(\int_{x_k}^{\alpha} \left(\int_{x_k}^{x} (x_k - s)^2 u^{(4)}(s) ds \right)^2 dx \right)^{1/2} \\
&\le C h_k^{1-m} \left(\int_{x_k}^{\alpha} \left(\int_{x_k}^{x} (x_k - s)^4 ds \int_{x_k}^{x} \left| u^{(4)}(s) \right|^2 ds \right) dx \right)^{1/2} \\
&\le C h_k^{1-m} \left(\int_{x_k}^{\alpha} \left(\int_{x_k}^{\alpha} (x_k - s)^4 ds \int_{x_k}^{\alpha} \left| u^{(4)}(s) \right|^2 ds \right) dx \right)^{1/2} \\
&\le C h_k^{4-m} \left| u \right|_{4,\mathring{e}_k^-} .
\end{aligned}$$

For the term of R_3^-, by formula (36i) given in Lemma 3.1, we note

$$\begin{aligned}
\left\| R_{3,1}^- \frac{d^m \psi_{k,3}^-}{dx^m} \right\|_{0,\mathring{e}_k^-} &= \left(\int_{x_k}^{\alpha} \left(\frac{1}{6} \int_x^{x_{k+1}} (x_{k+1} - s)^3 u^{(4)}(s) ds \frac{d^m \psi_{k,3}^-(x)}{dx^m} \right)^2 dx \right)^{1/2} \\
&\le \left(\int_{x_k}^{\alpha} \left(\frac{1}{6} \int_x^{\alpha} (x_{k+1} - s)^3 u^{(4)}(s) ds \frac{d^m \psi_{k,3}^-(x)}{dx^m} \right)^2 dx \right)^{1/2} \\
&\quad + \left(\int_{x_k}^{\alpha} \left(\frac{1}{6} \int_{\alpha}^{x_{k+1}} (x_{k+1} - s)^3 u^{(4)}(s) ds \frac{d^m \psi_{k,3}^-(x)}{dx^m} \right)^2 dx \right)^{1/2} \\
&\le C h_k^{-m} \left(\int_{x_k}^{\alpha} \left(\int_x^{\alpha} (x_{k+1} - s)^3 u^{(4)}(s) ds \right)^2 dx \right)^{1/2} \\
&\quad + C h_k^{-m} \left(\int_{x_k}^{\alpha} \left(\int_{\alpha}^{x_{k+1}} (x_{k+1} - s)^3 u^{(4)}(s) ds \right)^2 dx \right)^{1/2} \\
&\le C h_k^{-m} \left(\int_{x_k}^{\alpha} \left(\int_x^{\alpha} (x_{k+1} - s)^6 ds \int_x^{\alpha} \left| u^{(4)} \right|^2 (s) ds \right) dx \right)^{1/2} \\
&\quad + C h_k^{-m} \left(\int_{x_k}^{\alpha} \left(\int_{\alpha}^{x_{k+1}} (x_{k+1} - s)^6 ds \int_{\alpha}^{x_{k+1}} \left| u^{(4)}(s) \right|^2 ds \right) dx \right)^{1/2} \\
&\le C h_k^{4-m} \left(\left| u \right|_{4,\mathring{e}_k^-} + \left| u \right|_{4,\mathring{e}_k^+} \right),
\end{aligned}$$

$$\begin{aligned}
&\left\| R_{3,2}^- \frac{d^m \psi_{k,3}^-}{dx^m} \right\|_{0,\mathring{e}_k^-} \\
&= \left(\int_{x_k}^{\alpha} \left(\frac{1}{2} \left(-1 + \frac{\beta^-}{\beta^+} \right) (x_{k+1} - \alpha)^2 \int_x^{\alpha} (\alpha - s) u^{(4)}(s) ds \frac{d^m \psi_{k,3}^-(x)}{dx^m} \right)^2 dx \right)^{1/2} \\
&\le C h_k^{2-m} \left(\int_{x_k}^{\alpha} \left(\int_x^{\alpha} (\alpha - s) u^{(4)}(s) ds \right)^2 dx \right)^{1/2} \le C h_k^{4-m} \left| u \right|_{4,\mathring{e}_k^-} ,
\end{aligned}$$

$$
\begin{aligned}
&\left\| R_{3,3}^- \frac{d^m \psi_{k,3}^-}{dx^m} \right\|_{0,\mathring{e}_k^-} \\
&= \left(\int_{x_k}^{\alpha} \left(\frac{1}{6} \left(-1 + \frac{\beta^-}{\beta^+} \right) (x_{k+1} - \alpha)^3 \int_x^{\alpha} u^{(4)}(s) ds \frac{d^m \psi_{k,3}^-(x)}{dx^m} \right)^2 dx \right)^{1/2} \\
&\le C h_k^{-m} \left(\int_{x_k}^{\alpha} \left((x_{k+1} - \alpha)^3 \int_x^{\alpha} u^{(4)}(s) ds \right)^2 dx \right)^{1/2} \le C h_k^{4-m} |u|_{4,\mathring{e}_k^-} .
\end{aligned}
$$

Therefore, by the estimates above and $R_3^- = R_{3,1}^- + R_{3,2}^- + R_{3,2}^-$, we have

$$
\begin{aligned}
\left\| R_3^- \frac{d^m \psi_{k,3}^-}{dx^m} \right\|_{0,\mathring{e}_k^-} &\le \left\| R_{3,1}^- \frac{d^m \psi_{k,3}^-}{dx^m} \right\|_{0,\mathring{e}_k^-} + \left\| R_{3,2}^- \frac{d^m \psi_{k,3}^-}{dx^m} \right\|_{0,\mathring{e}_k^-} + \left\| R_{3,3}^- \frac{d^m \psi_{k,3}^-}{dx^m} \right\|_{0,\mathring{e}_k^-} \\
&\le C h_k^{4-m} \big(|u|_{4,\mathring{e}_k^-} + |u|_{4,\mathring{e}_k^+} \big).
\end{aligned}
$$

Now, let us estimate the term of R_4^-. By (36l) given in Lemma 3.1, we note

$$
\begin{aligned}
&\left\| R_{4,1}^- \frac{d^m \psi_{k,4}^-}{dx^m} \right\|_{0,\mathring{e}_k^-} \\
&= \left(\int_{x_k}^{\alpha} \left(\frac{1}{2} \int_x^{x_{k+1}} (x_{k+1} - s)^2 u^{(4)}(s) ds \frac{d^m \psi_{k,4}^-(x)}{dx^m} \right)^2 dx \right)^{1/2} \\
&\le \left(\int_{x_k}^{\alpha} \left(\frac{1}{2} \int_x^{\alpha} (x_{k+1} - s)^2 u^{(4)}(s) ds \frac{d^m \psi_{k,4}^-(x)}{dx^m} \right)^2 dx \right)^{1/2} \\
&\quad + \left(\int_{x_k}^{\alpha} \left(\frac{1}{2} \int_{\alpha}^{x_{k+1}} (x_{k+1} - s)^2 u^{(4)}(s) ds \frac{d^m \psi_{k,4}^-(x)}{dx^m} \right)^2 dx \right)^{1/2} \\
&\le C h_k^{1-m} \left(\int_{x_k}^{\alpha} \left(\int_x^{\alpha} (x_{k+1} - s)^2 u^{(4)}(s) ds \right)^2 dx \right)^{1/2} \\
&\quad + C h_k^{1-m} \left(\int_{x_k}^{\alpha} \left(\int_{\alpha}^{x_{k+1}} (x_{k+1} - s)^2 u^{(4)}(s) ds \right)^2 dx \right)^{1/2} \\
&\le C h_k^{4-m} \big(|u|_{4,\mathring{e}_k^-} + |u|_{4,\mathring{e}_k^+} \big),
\end{aligned}
$$

$$
\begin{aligned}
&\left\| R_{4,2}^{-} \frac{d^m \psi_{k,4}^{-}}{dx^m} \right\|_{0,\mathring{e}_k^-} \\
&= \left(\int_{x_k}^{\alpha} \left(\left(-1 + \frac{\beta^-}{\beta^+} \right) (x_{k+1} - \alpha) \int_x^{\alpha} (\alpha - s) u^{(4)}(s) ds \frac{d^m \psi_{k,4}^{-}(x)}{dx^m} \right)^2 dx \right)^{1/2} \\
&\leq C h_k^{2-m} \left(\int_{x_k}^{\alpha} \left(\int_x^{\alpha} (\alpha - s) u^{(4)}(s) ds \right)^2 dx \right)^{1/2} \\
&\leq C h_k^{2-m} \left(\int_{x_k}^{\alpha} \left(\int_x^{\alpha} (\alpha - s)^2 ds \int_x^{\alpha} \left| u^{(4)}(s) \right|^2 ds \right) dx \right)^{1/2} \leq C h_k^{4-m} |u|_{4,\mathring{e}_k^-},
\end{aligned}
$$

$$
\begin{aligned}
&\left\| R_{4,3}^{-} \frac{d^m \psi_{k,4}^{-}}{dx^m} \right\|_{0,\mathring{e}_k^-} \\
&= \left(\int_{x_k}^{\alpha} \left(\frac{1}{2} \left(-1 + \frac{\beta^-}{\beta^+} \right) (x_{k+1} - \alpha)^2 \int_x^{\alpha} u^{(4)}(s) ds \frac{d^m \psi_{k,4}^{-}(x)}{dx^m} \right)^2 dx \right)^{1/2} \\
&\leq C h_k^{3-m} \left(\int_{x_k}^{\alpha} \left(\int_x^{\alpha} u^{(4)}(s) ds \right)^2 dx \right)^{1/2} \\
&\leq C h_k^{3-m} \left(\int_{x_k}^{\alpha} \left(\int_x^{\alpha} 1^2 ds \int_x^{\alpha} \left| u^{(4)}(s) \right|^2 ds \right) dx \right)^{1/2} \leq C h_k^{4-m} |u|_{4,\mathring{e}_k^-}.
\end{aligned}
$$

Therefore, by the estimates above and $R_4^- = R_{4,1}^- + R_{4,2}^- + R_{4,3}^-$, we have

$$
\begin{aligned}
\left\| R_4^- \frac{d^m \psi_{k,4}^-}{dx^m} \right\|_{0,\mathring{e}_k^-} &\leq \left\| R_{4,1}^- \frac{d^m \psi_{k,4}^-}{dx^m} \right\|_{0,\mathring{e}_k^-} + \left\| R_{4,2}^- \frac{d^m \psi_{k,4}^-}{dx^m} \right\|_{0,\mathring{e}_k^-} + \left\| R_{4,3}^- \frac{d^m \psi_{k,4}^-}{dx^m} \right\|_{0,\mathring{e}_k^-} \\
&\leq C h_k^{4-m} \big(|u|_{4,\mathring{e}_k^-} + |u|_{4,\mathring{e}_k^+} \big).
\end{aligned}
$$

Using these estimates in (42), we have

$$
\begin{aligned}
\left\| \big(I_{h,k} u\big)^{(m)} - u^{(m)} \right\|_{0,\mathring{e}_k^-} &\leq \sum_{j=1}^{4} \left\| R_j^- \frac{d^m \psi_{k,j}^-}{dx^m} \right\|_{0,\mathring{e}_k^-} \\
&\leq C h_k^{4-m} \big(|u|_{4,\mathring{e}_k^-} + |u|_{4,\mathring{e}_k^+} \big), \ m = 0,1,2,3. \qquad (44)
\end{aligned}
$$

By Lemma 2.1, Lemma 3.2, and similar arguments, we can show that

$$
\begin{aligned}
\left\| \big(I_{h,k} u\big)^{(m)} - u^{(m)} \right\|_{0,\mathring{e}_k^+} &\leq \sum_{j=1}^{4} \left\| R_j^+ \frac{d^m \psi_{k,j}^+}{dx^m} \right\|_{0,\mathring{e}_k^+} \\
&\leq C h_k^{4-m} \big(|u|_{4,\mathring{e}_k^-} + |u|_{4,\mathring{e}_k^+} \big), \ m = 0,1,2,3. \qquad (45)
\end{aligned}
$$

Finally, estimate (43) follows easily from (44) and (45).

■

By the estimates in (35) and (43) and standard arguments, we derive the estimates for the error in the IFE interpolation globally over the whole solution domain in the following theorem.

Theorem 3.2. *There exists a constant C independent of the interface location α such that for every $u \in \tilde{H}^4(\Omega)$ satisfying the jump conditions given in* (2c)*, we have the following estimates for the IFE interpolation defined by* (34)*:*

$$\left\|(I_h u)^{(m)} - u^{(m)}\right\|_{0,\Omega} \le Ch^{4-m}\big(\,|u|_{4,\Omega^-} + |u|_{4,\Omega^+}\,\big),\ m = 0, 1, 2, 3. \tag{46}$$

4. Error Bounds for the IFE solution

In this section, we derive the error bounds for the IFE solution for the interface problem (2), which, as usual, can be assumed to have homogeneous boundary conditions, i.e., $w_0 = w_1 = w_2 = w_3 = 0$. Formally, we multiply (2a) by a test function $v \in H_0^2(\Omega)$, and then integrate both sides on the domain Ω. A direct application of integration by parts and jump conditions in (2c) gives

$$\int_0^1 \beta u''(x)v''(x)dx = \int_0^1 fvdx.$$

Define the bilinear form $a(\cdot,\cdot) : H_0^2(\Omega) \times H_0^2(\Omega) \to \mathbb{R}$:

$$a(u, v) = \int_0^1 \beta u''(x)v''(x)dx. \tag{47}$$

Then, we obtain the weak form of the beam interface problem: find $u \in H_0^2(\Omega)$ such that

$$a(u, v) = (f, v),\ \forall v\ \in H_0^2(\Omega). \tag{48}$$

Since the mapping $u \to \|u''\|_{0,\Omega}$ is a norm over the space $H_0^2(\Omega)$, equivalent to the norm $\|\cdot\|_{2,\Omega}$, it is clear that $a(u, u) \ge C_1 \|u\|_{2,\Omega}^2$ which implies that the bilinear form $a(\cdot,\cdot)$ is coercive. On the other hand, it is obvious that the following continuity of the bilinear form holds:

$$a(u, v) \le C_2 |u|_{2,\Omega} |v|_{2,\Omega} \le C_3 \|u\|_{2,\Omega} \|v\|_{2,\Omega}\,. \tag{49}$$

Therefore, the weak problem of the beam interface problem has a unique solution according to the Lax-Milgram theorem.

We use the finite dimensional IFE space $S_h(\Omega) \subset H^2(\Omega)$ to discretize the weak formulation (48) such that the cubic Hermite IFE solution $u_h \in S_{h,0}(\Omega)$ to the interface problem described by (2) is defined by

$$a(u_h, v_h) = (f, v_h),\ \forall v_h\ \in S_{h,0}(\Omega), \tag{50}$$

here, $S_{h,0}(\Omega) = \{v \in S_h(\Omega) \mid v(0) = v(1) = v'(0) = v'(1) = 0\}$. Applying the Lax-Milgram theorem again, we can see that the IFE solution u_h uniquely exists, and we can show that u_h converges optimally to the exact solution u of the interface problem (2) in the following theorem.

Theorem 4.1. *Assume that the exact solution u to the interface problem described by* (2) *is a function in $\tilde{H}^4(\Omega)$. Then, there exists a constant C independent of the interface location such that the IFE solution u_h has the following error bounds:*

$$|u_h - u|_{m,\Omega} \le Ch^{4-m}\big(\,|u|_{4,\Omega^-} + |u|_{4,\Omega^+}\,\big),\ m = 0, 2. \tag{51}$$

Proof. Since $S_{h,0}(\Omega) \subset H_0^2(\Omega)$, then, by (48) and (50), we have

$$a(u - u_h, v_h) = 0,\ \forall\, v_h \in S_{h,0}(\Omega).$$

Hence, by the coercivity,

$$C_1|u_h-u|_{2,\Omega}^2 \le a(u_h-u,u_h-u) = a(u_h-u,v_h-u) \le C_2|u_h-u|_{2,\Omega}|v_h-u|_{2,\Omega},\ \forall\, v_h\in S_{h,0}(\Omega)$$

which, according to Theorem 3.2, implies that

$$|u_h-u|_{2,\Omega} \le C|I_h u-u|_{2,\Omega} \le Ch^2\big(\,|u|_{4,\Omega^-}+|u|_{4,\Omega^+}\,\big), \tag{52}$$

where $I_h u \in S_{h,0}(\Omega)$ is the IFE interpolation of u. The adjoint problem of (48) is: Given any $g\in L^2(\Omega)$, find an element $\phi_g \in H_0^2(\Omega)$ such that

$$a(\phi_g,v)=(g,v),\ \forall\, v\in H_0^2(\Omega).$$

It is easy to see that the adjoint problem is regular, i.e., $\phi_g|_{\Omega^\pm}\in \tilde{H}^4(\Omega)$ and there exists a constant C such that

$$|\phi_g|_{4,\Omega^-}+|\phi_g|_{4,\Omega^+} \le C\,\|g\|_{0,\Omega}\,.$$

Then, by Theorem 3.2 again, we have

$$\inf_{\phi_h\in S_{h,0}(\Omega)}|\phi_g-\phi_h|_{2,\Omega} \le |\phi_g-I_h\phi_g|_{2,\Omega} \le Ch^2|\phi_g|_{4,\Omega} \le Ch^2\,\|g\|_{0,\Omega}\,. \tag{53}$$

Since $u-u_h\in H_0^2(\Omega)$, we have

$$a(\phi_g,u-u_h)=(g,u-u_h).$$

On the other hand, we have $a(u-u_h,\phi_h)=0,\ \forall\,\phi_h\in S_{h,0}(\Omega)$. Combining these two relations, we obtain

$$|(g,u-u_h)| = |a(u-u_h,\phi_g-\phi_h)| \le C_2|u-u_h|_{2,\Omega}|\phi_g-\phi_h|_{2,\Omega},$$

which leads to

$$|(g,u-u_h)| \le C|u-u_h|_{2,\Omega}\inf_{\phi_h\in S_{h,0}(\Omega)}|\phi_g-\phi_h|_{2,\Omega} \le Ch^2\,\|g\|_{0,\Omega}\,|u-u_h|_{2,\Omega}$$

because of (53). Using the above inequality and (52), we have

$$\|u-u_h\|_{0,\Omega} \le \sup_{g\in L^2(\Omega)}\frac{|(g,u-u_h)|}{\|g\|_{0,\Omega}} \le Ch^4\big(\,|u|_{4,\Omega^-}+|u|_{4,\Omega^+}\,\big), \tag{54}$$

and estimates established in (52) and (54) together lead to (51).

■

We say that a set of meshes $\mathcal{T}_h, h>0$ of Ω is quasi-uniform provided that there exists a constant C such that

$$\frac{h}{\min_{e_k\in\mathcal{T}_h} h_k} \le C,\ \ h>0. \tag{55}$$

For the IFE solution u_h produced with a set of quasi-uniform meshes $\mathcal{T}_h, h>0$, we can further derive its optimal error bounds in the semi-H^1 and semi-H^3 norms in the following theorem.

Theorem 4.2. *Assume that the exact solution u to the interface problem described by (2) is a function in $\tilde{H}^4(\Omega)$ and the IFE solution u_h is generated on a set of quasi-uniform meshes. Then, there exists a constant C independent of the interface location such that:*

$$|u_h-u|_{m,\Omega} \le Ch^{4-m}\big(\,|u|_{4,\Omega^-}+|u|_{4,\Omega^+}\,\big),\ m=1,3. \tag{56}$$

Proof. By the triangle inequality, we have

$$|u_h - u|_{m,\Omega} \le |u_h - I_h u|_{m,\Omega} + |I_h u - u|_{m,\Omega}\,,\ m = 1, 3. \tag{57}$$

By Lemma 2.2, the standard inverse inequalities on non-interface elements, and the quasi-uniform property (55), we have

$$\begin{aligned}
|u_h - I_h u|_{m,\Omega} &\le C \sum_{e_k \in \mathcal{T}_h} |u_h - I_h u|_{m,\mathring{e}_k} \le \sum_{e_k \in \mathcal{T}_h} C h_k^{-1} |u_h - I_h u|_{m-1,\mathring{e}_k} \\
&\le \sum_{e_k \in \mathcal{T}_h} C h_k^{-1} (|u_h - u|_{m-1,\mathring{e}_k} + |u - I_h u|_{m-1,\mathring{e}_k}) \\
&\le C h^{-1} (|u_h - u|_{m-1,\Omega} + |u - I_h u|_{m-1,\Omega}),\ m = 1, 3.
\end{aligned}$$

Then, by Theorem 3.2 and Theorem 4.1, we have

$$|u_h - I_h u|_{m,\Omega} \le C h^{4-m} (|u|_{4,\Omega^-} + |u|_{4,\Omega^+}),\ m = 1, 3.$$

Finally, applying this inequality and Theorem 3.2 to (57), we obtain

$$|u_h - u|_{m,\Omega} \le C h^{4-m} (|u|_{4,\Omega^-} + |u|_{4,\Omega^+}),\ m = 1, 3, \tag{58}$$

and (56) is proved.

∎

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