DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS–SERIES B
Volume **7**, Number **4**, June **2007**

Website: http://AIMsciences.org

pp. **807–823**

# ERROR ESTIMATION OF A CLASS OF QUADRATIC IMMERSED FINITE ELEMENT METHODS FOR ELLIPTIC INTERFACE PROBLEMS

Tao Lin

Department of Mathematics
Virginia Tech, Blackburg, VA, 24061-0123

Yanping Lin

Department of Mathematical and Statistics Science
University of Alberta, Edmonton AB, T6G 2G1, Canada

Weiwei Sun

Department of Mathematics
The City University of Hong Kong
Koloon Tong, Hong Kong, China

Abstract. We carry out error estimation of a class of immersed finite element (IFE) methods for elliptic interface problems with both perfect and imperfect interface jump conditions. A key feature of these methods is that their partitions can be independent of the location of the interface. These quadratic IFE spaces reduce to the standard quadratic finite element space when the interface is not in the interior of any element. More importantly, we demonstrate that these IFE spaces have the optimal (slightly lower order in one case) approximation capability expected from a finite element space using quadratic polynomials.

## 1. Introduction.

The main purpose of this article is to carry out error estimation of a class of quadratic immersed finite element (IFE) methods for elliptic interface problems: find a function $u(x)$ such that

$$-\left(\beta(x)u'\right)' = f, \; x \in (0,1), \tag{1.1}$$

$$u(0) = u_0, u(1) = u_1, \tag{1.2}$$

$$[u]|_{x=\alpha} = A, \quad [\beta u']|_{x=\alpha} = B \quad \text{at} \quad x = \alpha, \tag{1.3}$$

where the solution domain $\Omega = (0,1)$ is separated by the interface $x = \alpha$ into two sub-domains, the coefficient $\beta(x)$ is a piecewise smooth function such that

$$\beta(x) = \begin{cases} \beta^-(x), & x \in (0,\alpha), \\ \beta^+(x), & x \in (\alpha,1). \end{cases}$$

The methods and their related analysis considered here can be easily extended to more sophisticated cases in which, for example, there might be multiple interfaces, and the involved differential equation might contain certain nonlinearity. The need

2000 *Mathematics Subject Classification.* 90A09, 65K10, 65M12, 65M60.
*Key words and phrases.* elliptic, interface, jump condition, immersed finite element method, error estimates.

to efficiently solve elliptic interface problems appears in many applications, see [6, 11, 22, 18, 27, 28] for only a few specific examples.

Numerous methods, such as finite difference (FD) methods [16, 30, 33], finite element (FE) method [2], and collocation methods [29], have been developed to solve elliptic interface problems. As proposed by [3], a general idea of solving interface problems or boundary value problems with rough coefficients is to use a finite element space constructed specifically according to the problem to be solved. One way to implement this idea in a finite element method is to align element edges of the partition along the interface (the so called boundary conforming partition), and consequently the resulted finite element method can produce approximation with an optimal convergence rate [2, 7, 10].

However, the finite element methods based on boundary conforming partitions are awkward for those applications such as the optimal shape design problem [13, 15] in which an interface problem has to be solved repeatedly, each time with a different interface $\Gamma$ (either due to variation in its shape or position) because all of those quantities involving the partition have to be generated over and over again. Also, there are many applications such as those in [27, 32] that require to solve interface problems efficiently over a structured (preferably a Cartesian) partition. For these types of applications, [4] implements the fundamental idea of [3] using linear polynomials with a partition independent of the interface. This idea has been further developed as the so called immersed finite element methods [8, 21, 23, 24, 25, 26].

The main differences between the IFE methods and the standard FE methods for interface problems can be summarized as follows:

- To achieve the optimal accuracy, the partition used in a standard FE method has to be formed according to the location of the interface, but the partition of an IFE method can be formed independently of the interface.
- On the other hand, the basis functions in a standard FE method are formed independent of the interface, but some of the basis functions in an IFE method will incorporate the interface location and the interface jump conditions.

There have been publications about IFE spaces using linear [4, 23, 24, 25] and bilinear [26] polynomials. A class of quadratic IFE spaces have been introduced only recently [8] from which we can see that these quadratic IFE space have many desirable features such as

- They have the partition of unit of the local nodal basis functions.
- They are consistent with the standard quadratic finite element space in the sense that they reduce to the standard quadratic FE space when the discontinuity in the coefficient disappears or when the discontinuity happens on the edges of the elements in the partition.

Moreover, the extensive numerical data presented in [8] suggest that these quadratic IFE spaces have the optimal or slightly lower order approximation capability expected from a finite element space using quadratic polynomials. We point out that a class of immersed elements has been proposed in [3] and quadratic elements in this paper are very different from [3]. In [3] the immersed elements are constructed by jump continuities and moments (related to the mixed finite elements), but those in this paper are constructed using high order jump continuity or hierarchically based on the linear elements. As long as linear IFE is concerned the approaches in [3] and ours below are equivalent and thus generate the same linear elements, but the error analysis and numerical implementations are different.

Recently, the authors of [9] considered a spherical interfaces dynamos modeling where the interfaces are known spheres in the universe, and our 3D linear IFE constructed in [21] can be extended to this case to provide a useful alternative approach for such important problems in astrophysics. The authors of [5] studied a simple PDE model for a pulsed amperometric ion working mechanism, where the quadratic jump interface conditions are derived, the solution decomposition and numerical method are provided in [17], our IFE here can also be employed in such case to give a more accurate numerical solution techniques. The so-called WPE method [31] is proposed for Fokker-Planck equations in bimolecular transport process with interfaces when the motor potential is discontinuous function, IFE discussed below can also be used for such equation with a proper homogeneity. Recently higher order IFE with mixed setting for 1D elliptic problems has been studied [1].

Our goal here is to present the pertinent analysis to theoretically confirm our numerical results about the accuracy of these quadratic IFE spaces presented in [8]. In addition, we also discuss how to apply IFE methods to solve more complicated interface problems such as those with imperfect interface jump conditions.

This paper is organized as follows. In Section 2, we derive the error estimates in Sobolev norms for the interpolant in these quadratic IFE spaces. Section 3 is dedicated to the applications of the IFE spaces to solve standard elliptic interface problems and their extensions. We use the estimates in Section 2 to derive the error estimates for the IFE solutions for the elliptic interface problems. The results in both Sections 2 and 3 confirm that the IFE solutions to the elliptic interface problems considered here definitely have the optimal (or slightly lower in one case) approximation capability expected from a finite element space using quadratic polynomials. As specific examples to show that the IFE spaces discussed here can be applied to other problems, we will discuss interface problems with non-homogeneous jump conditions, interface problems with imperfect interface jump conditions.

## 2. **Errors bounds for the IFE interpolants.**

Since the accuracy of the finite element solution is closely related with the accuracy of the interpolant constructed in the finite element space used, it is important to derive the estimates for the errors in the IFE interpolants. Without loss of generality, our discussion focuses on the IFE spaces for the interface problems with homogeneous jump conditions, i.e., $A = B = 0$ in (1.3). All the results obtained can be extended to the cases in which the jump conditions are non-homogeneous, and the related discussions are presented in Section 3.

Deriving the error estimates for the IFE spaces in the $H^1$ and $L^2$ Sobolev norms are the main task of this section. From now on, we use the following modified Sobolev spaces for those functions satisfying homogeneous jump conditions at the interface: for $k \geq 1$, we let

$$H^k_\alpha(\Omega) \quad = \quad \{v \in L^2(\Omega) \mid v \in H^k(0,\alpha) \cap H^k(\alpha,1), \qquad [v]_{x=\alpha} = [\beta v']_{x=\alpha} = 0\},$$

with the norms defined by

$$||v||_{k,\alpha} = \left(||v||^2_{H^k(0,\alpha)} + ||v||^2_{H^k(\alpha,1)}\right)^{1/2}, \qquad k \geq 0.$$

The related space $H^k_{0,\alpha}(\Omega)$ formed by those functions of $H^k_\alpha(\Omega)$ with zero boundary values can be defined as usual. We will also use $\|\cdot\|_r$ to denote the norm of the usual Sobolev space $H^r(S)$ defined over a set $S$.

2.1. **The linear IFE space.** In this section we consider the interpolation accuracy of the IFE space based on linear polynomials. The original implementation in two dimension can be found in [4] and its one dimensional version is discussed in [23]. Both of these article present some preliminary analysis on this IFE space. Our goal here is to carry out the error estimation on the interpolation error of this IFE in the Sobolev norms suitable for deriving error estimates of the related IFE solution for interface problems.

First, let us recall the definition of the one dimensional linear IFE space [23]. Without loss of generality, let us consider a uniform partition of $\bar\Omega = [0,1]$:

$$\begin{aligned} &0 = x_0 < x_1 < \cdots < x_N = 1, \\ &h_i = x_i - x_{i-1} = h,\ i = 1, 2, \cdots, N, \\ &\mathcal{T}_h = \cup_{i=0}^{N-1} e_i = \bar\Omega,\ e_i = [x_i, x_{i+1}],\ i = 0, 1, 2, \cdots, N-1. \end{aligned} \tag{2.1}$$

At the node $x_j$, we define a piecewise linear function $\phi_j(x)$ such that

$$\phi_j(x_i) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases} \quad \text{and } [\phi_j]_{x=\alpha} = [\beta\phi_j']_{x=\alpha} = 0.$$

It is easy to see that function $\phi_j(x)$ is uniquely determined by the conditions above. Then the linear IFE space is defined as

$$S_h^1(\Omega) = span\{\phi_j, j = 0, 1, \cdots, N\}.$$

Since it is obvious that $\phi_j \in H^1_\alpha(\Omega)$, we have $S_h^1(\Omega) \subset H^1_\alpha(\Omega)$.

We call the element $e_j$ a non-interface element if $\alpha \notin (x_j, x_{j+1})$; otherwise we call it an interface element. The definition of the linear IFE space $S_h^1(\Omega)$ implies that on each non-interface element we use the usual linear polynomials, but on the interface element $e_j$ we use piecewise linear polynomials defined by the sub-intervals $(x_j, \alpha)$ and $(\alpha, x_{j+1})$.

Now, for any $u \in C^0([0,1])$, we consider its interpolation in $S_h^1(\Omega)$:

$$I_h u(x) = \sum_{i=0}^{N} u(x_i)\phi_i(x).$$

First, we can easily show that $I_h u$ can be represented in terms of the linear Lagrange cardinal polynomials on each element.

**Lemma 1.** *The interpolation operator $I_h$ is given for $\alpha \in (x_j, x_{j+1})$ by*

$$I_h u(x) = \begin{cases} -u(x_j)L_{j+1}(x) + u(x_{j+1})L_j(x), & \alpha \notin [x_j, x_{j+1}],\ \textit{for all } j, \\ u(x_j)L_{j,0}(x) + u_{h,\alpha}L_{j,\alpha}(x), & x \in [x_j, \alpha], \\ u_{h,\alpha}L_{j+1,\alpha}(x) + u(x_{j+1})L_{j+1,1}(x), & x \in [\alpha, x_{j+1}]. \end{cases} \tag{2.2}$$

*where $u_{h,\alpha}$ is the value determined by the interface jump conditions and is given by*

$$u_{h,\alpha} = \Delta^{-1}(\beta^+ u_{j+1} L'_{j+1,1}(\alpha) - \beta^- u_j L'_{j,0}(\alpha)),$$

*with*

$$\begin{aligned}
\Delta &= \beta^- L'_{j,\alpha} - \beta^+ L'_{j+1,\alpha} = \frac{\beta^-}{\alpha - x_j} + \frac{\beta^+}{x_{j+1} - \alpha} > 0, \\
L_{j,0}(x) &= \frac{x-\alpha}{x_j - \alpha}, \qquad L_{j,\alpha}(x) = \frac{x - x_j}{\alpha - x_j}, \\
L_{j+1,\alpha}(x) &= \frac{x - x_{j+1}}{\alpha - x_{j+1}}, \qquad L_{j+1,1}(x) = \frac{x-\alpha}{x_{j+1}-\alpha}, \qquad \textit{and} \qquad L_j(x) = \frac{x - x_j}{h}.
\end{aligned}$$

□

With this lemma, we can show that the linear IFE space has the usual approximation capability expected from a finite element space using linear polynomials.

**Theorem 1.** *For any $u \in H^2_\alpha(\Omega)$, there exists a positive constant $C > 0$, independent of $h$ and $u$ such that*

$$||u - I_h u||_{0,\alpha} + h||u - I_h u||_{1,\alpha} \quad \leq \quad C\rho h^2 ||u''||_{0,\alpha}. \tag{2.3}$$

*where $\rho = \max\{\beta^-/\beta^+, \beta^+/\beta^-\}$.*

*Proof.* Let $\tilde{I}_{h,\alpha}$ be the standard Lagrange linear interpolation operator defined on the partition formed by nodes $x_i, i = 0, 1, \cdots, N$ and additional node $x = \alpha$. Thus $I_h u(x)$ and $\tilde{I}_{h,\alpha} u(x)$ are identical except on the interface element $e_j = [x_j, x_{j+1}]$.

Since $u - I_h u = (u - \tilde{I}_{h,\alpha} u) + (\tilde{I}_{h,\alpha} u - I_h u)$, it is sufficient to estimate $(\tilde{I}_{h,\alpha} u - I_h u)$ on the interface element only. It is easy to see that

$$\tilde{I}_{h,\alpha} u - I_h u = \begin{cases} 0, & x \notin (x_j, x_{j+1}), \\ (u(\alpha) - u_{h,\alpha}) L_{j,\alpha}(x), & x \in [x_j, \alpha], \\ (u(\alpha) - u_{h,\alpha}) L_{j+1,\alpha}(x), & x \in [\alpha, x_{j+1}]. \end{cases} \tag{2.4}$$

Thus we only need to estimate the difference $u(\alpha) - u_{h,\alpha}$. By a simple calculation and the interface jump conditions of $u$ we find that

$$u(\alpha) - u_{h,\alpha} = \Delta^{-1}\{\beta^-(\tilde{I}_{h,\alpha} u - u)'(\alpha^-) - \beta^+(\tilde{I}_{h,\alpha} u - u)'(\alpha^+)\}.$$

Let $e = \tilde{I}_{h,\alpha} u - u$, then $e(x_j) = e(\alpha) = e(x_{j+1}) = 0$, we thus see that there exist $\xi \in (x_j, \alpha)$ and $\eta \in (a, x_{j+1})$ such that

$$\begin{aligned}
|(\tilde{I}_{h,\alpha} u - u)'(\alpha^-)| &= \left| \int_\alpha^\xi e''(x) dx \right| \leq (\alpha - x_j)^{1/2} \left( \int_{x_j}^\alpha |u''|^2 dx \right)^{1/2}, \\
|(\tilde{I}_{h,\alpha} u - u)'(\alpha^+)| &= \left| \int_\alpha^\eta e''(x) dx \right| \leq (x_{j+1} - \alpha)^{1/2} \left( \int_\alpha^{x_{j+1}} |u''|^2 dx \right)^{1/2}.
\end{aligned}$$

Hence, by noticing that

$$\Delta^{-1} = \frac{(\alpha - x_j)(x_{j+1} - \alpha)}{\beta^-(x_{j+1} - \alpha) + \beta^+(\alpha - x_j)} \leq \frac{(\alpha - x_j)(x_{j+1} - \alpha)}{\min\{\beta^-, \beta^+\} h},$$

it follows from the above two inequalities that

$$|u(\alpha) - u_{h,\alpha}| \leq C h^{3/2} \left( \int_{x_j}^\alpha |u''|^2 dx + \int_\alpha^{x_{j+1}} |u''|^2 dx \right)^{1/2},$$

which and (2.4) imply that

$$\begin{aligned} & ||\tilde{I}_{h,\alpha}u - I_h u||_{0,\alpha} + h||\tilde{I}_{h,\alpha}u - I_h u||_{1,\alpha} \\ \leq \; & Ch^2 \left( \int_{x_j}^{\alpha} |u''|^2 dx + \int_{\alpha}^{x_{j+1}} |u''|^2 dx \right)^{1/2} . \end{aligned}$$

Finally the proof can be completed from the above inequality, the standard error estimates of $\tilde{I}_h u$, and triangle inequality. □

2.2. **A hierarchical quadratic IFE space.** We now consider the interpolation accuracy of the quadratic IFE space that is hierarchically formed by multiplying two linear IFE functions satisfying the interface jump conditions, see [8] for more details and basic properties of this quadratic IFE space. The extensive numerical experiments reported in [8] indicates that this IFE space does not have the optimal order of accuracy associated with quadratic polynomials, but data there strongly suggest that the order of accuracy of this IFE space seems to be $O(h^{5/2})$ in the $L^2$ norm and $O(h^{3/2})$ in the $H^2$ norm. The analysis in this section gives a positive answer to these expectations.

Consider the partition $\mathcal{T}_h$ of $\Omega$ defined by (2.1) in which each element $e_k$ has three nodes:

$$t_{k,1} = x_k, \; t_{k,2} = x_{k+1/2} = \frac{x_k + x_{k+1}}{2}, \; t_{k,3} = x_{k+1}.$$

Assume that $e_j$ is the interface element. On a non-interface element $e_k$, we let $\psi_{k,i}(x), i = 1,2,3$ be the quadratic polynomials such that $\psi_{k,i}(t_{k,i}) = 1, \psi_{k,i}(t_{k,l}) = 0, l \neq k$, and let

$$S_h^2(e_k) = span\{\psi_{k,1}, \; \psi_{k,2}, \; \psi_{k,3}\}.$$

On the interface element $e_j$, we let $\tilde{\psi}_{j,i}(x), i = 1,2,3$ be the piecewise quadratic polynomials hierarchically formed from the linear IFE functions as follows:

$$\tilde{\psi}_{j,1}(x) = l_{1,2}(x) l_{1,3}(x), \; \tilde{\psi}_{j,2}(x) = l_{2,1}(x) l_{2,3}(x), \; \tilde{\psi}_{j,3}(x) = l_{3,1}(x) l_{3,2}(x). \quad (2.5)$$

where

$$l_{i,j}(x) = \begin{cases} a_1 x + a_0, & x < \alpha, \\ b_1 x + b_0, & x \geq \alpha, \end{cases} \quad \begin{cases} l_{i,j}(t_{k,i}) = 1, \; l_{i,j}(t_{k,j}) = 0, \\ \\ [l_{i,j}]_{x=\alpha} = 0, \; [\beta l'_{i,j}]_{x=\alpha} = 0. \end{cases} \quad (2.6)$$

We now let

$$\tilde{S}_h^2(e_j) = span\{\tilde{\psi}_{j,1}, \; \tilde{\psi}_{j,2}, \; \tilde{\psi}_{j,3}\}.$$

The hierarchical quadratic IFE space $\tilde{S}_h(\Omega)$ is then defined as follows: $v \in \tilde{S}_h(\Omega)$ if

- $v \in C(\bar{\Omega})$.
- $v|_{e_k} \in S_h^2(e_k), \; k \neq j, \; v|_{e_j} \in \tilde{S}_h^2(e_j)$.

By simple calculations we can see that $v|_{e_j} \in \tilde{S}_h^2(e_j)$ is uniquely determined by its values at $t_{j,i}, i = 1,2,3$ and the interface jump conditions:

$$[v]_{x=\alpha} = [\beta v']_{x=\alpha} = [\beta^2 v'']_{x=\alpha} = 0.$$

Please refer to [8] for more details and basic properties of this quadratic IFE space.

Without loss of generality, we assume that $x_j < x_{j+1/2} < \alpha < x_{j+1}$. For any $u \in H_\alpha^3(\Omega)$, we let $I_h u(x)$ be its interpolant in $\tilde{S}_h(\Omega)$.

We now consider an auxiliary partition $\tilde{\mathcal{T}}_h$ whose elements are formed by $x_i, i = 0, 2, \cdots, N$ and $x = \alpha$ together with the middle nodes $x_{i+1/2}, i = 0, 1, \cdots, N-1$ in each of the elements except for the elements $(x_j, \alpha)$ and $(\alpha, x_{j+1})$. We then introduce a Lagrange-Hermite interpolant of $u$ on $\tilde{\mathcal{T}}_h$ as follows. On a non-interface element $e_i$ of $\tilde{\mathcal{T}}_h$, we let $\tilde{I}_{h,\alpha}u(x)$ be the usual Lagrange quadratic interpolant of $u$ defined by the nodal points of $e_i$. On the two elements adjacent to the interface point $\alpha$, we let

$$\tilde{I}_{h,\alpha}u(x) = \begin{cases} u_j L_j(x) + u_{j+1/2}L_{j+1/2}(x) + u(\alpha)L_\alpha(x), & x \in (x_j, \alpha), \\ u(\alpha)H_{\alpha,0}(x) + u'(\alpha^+)H_{\alpha,1}(x) + u_{j+1}H_{j+1}(x), & x \in (\alpha, x_{j+1}), \end{cases} \tag{2.7}$$

where

$$\begin{aligned} L_j(x) &= \frac{(x - x_{j+1/2})(x-\alpha)}{(x_j - x_{j+1/2})(x_j - \alpha)}, \qquad L_{j+1/2}(x) = \frac{(x-x_j)(x-\alpha)}{(x_{j+1/2} - x_j)(x_{j+1/2} - \alpha)}, \\ L_\alpha(x) &= \frac{(x - x_{j+1/2})(x - x_j)}{(\alpha - x_{j+1/2})(\alpha - x_j)}, \qquad H_{j+1} = \frac{(x-\alpha)^2}{(x_{j+1/2} - \alpha)^2}, \\ H_{\alpha,0}(x) &= 2\frac{(x - x_{j+1})}{(\alpha - x_{j+1})} - \frac{(x - x_{j+1})^2}{(\alpha - x_{j+1})^2}, \qquad H_{\alpha,1}(x) = \frac{(x-\alpha)(x - x_{j+1})}{(\alpha - x_{j+1})}. \end{aligned}$$

For the above polynomials, we have the following estimates:

$$\int_{x_j}^{\alpha} L_\alpha^2(x)dx \le C\frac{h^3}{(\alpha - x_{j+1/2})^2}, \ \int_{x_j}^{\alpha} (L'_\alpha(x))^2 dx \le \frac{Ch}{(\alpha - x_{j+1/2})^2}. \tag{2.8}$$

$$\int_{\alpha}^{x_{j+1}} (H_{\alpha,0}(x))^2 dx = \frac{8}{15}(x_{j+1} - \alpha), \ \int_{\alpha}^{x_{j+1}} (H_{\alpha,1}(x))^2 dx = \frac{1}{30}(x_{j+1} - \alpha)^3, \tag{2.9}$$

$$\int_{\alpha}^{x_{j+1}} (H'_{\alpha,0}(x))^2 dx = \frac{4}{3}\frac{1}{x_{j+1} - \alpha}, \ \int_{\alpha}^{x_{j+1}} (H'_{\alpha,1}(x))^2 dx = \frac{1}{3}(x_{j+1} - \alpha). \tag{2.10}$$

**Lemma 2.** *The quadratic IFE interpolant $I_h u(x) \in \tilde{S}_h(\Omega)$ has the following representation:*

$$I_h u(x) = \begin{cases} \tilde{I}_{h,\alpha}u(x), & x \notin (x_j, x_{j+1}), \\ u_j L_j(x) + u_{j+1/2}L_{j+1/2}(x) + \bar{u}_\alpha L_\alpha(x), & x \in (x_j, \alpha), \\ \bar{u}_\alpha H_{\alpha,0}(x) + \bar{u}'_\alpha H_{\alpha,1}(x) + u_{j+1}H_{j+1}(x), & x \in (\alpha, x_{j+1}). \end{cases} \tag{2.11}$$

*where $\bar{u}_\alpha$ and $\bar{u}'_\alpha$ are defined by*

$$\begin{aligned} \bar{u}_\alpha &= \Delta_5^{-1}\{\beta^-(\beta^+)^2 H_{\alpha,1}{}''(\alpha)[u_j L'_j(\alpha) + u_{j+1/2}L'_{j+1/2}(\alpha)] \qquad (2.12) \\ &\quad + \beta^+[(\beta^+)^2 u_{j+1}H''{}_{j+1}(\alpha) - (\beta^-)^2(u_j L''{}_j(\alpha) + u_{j+1/2}L''{}_{j+1/2}(\alpha))]\}, \\ \bar{u}'_\alpha &= \frac{1}{(\beta^+)^2 H''{}_{\alpha,1}(\alpha)}\left\{(\beta^-)^2[u_j L''{}_j(\alpha) + u_{j+1/2}L''{}_{j+1/2}(\alpha) + \bar{u}_\alpha L''{}_\alpha(\alpha)]\right. \\ &\quad \left. -(\beta^+)^2[\bar{u}_\alpha H''{}_{\alpha,0}(\alpha) + u_{j+1}H''{}_{j+1}(\alpha)]\right\}, \qquad (2.13) \\ \Delta_5 &= -\beta^-(\beta^+)^2 L'_\alpha(\alpha)H''{}_{\alpha,1}(\alpha) + \beta^+[(\beta^-)^2 L''{}_\alpha(\alpha) - (\beta^+)^2 H''{}_{\alpha,0}(\alpha)]. \qquad (2.14) \end{aligned}$$

*Proof.* Applying the jump conditions $[\beta(I_h u)']_{x=\alpha} = 0$ and $[\beta^2(I_h u)'']_{x=\alpha} = 0$ we have

$$\beta^-\{u_j L'_j(\alpha) + u_{j+1/2}L'_{j+1/2}(\alpha) + \bar{u}_\alpha L'_\alpha(\alpha)\} = \beta^+\bar{u}'_\alpha, \tag{2.15}$$

$$\begin{aligned} &(\beta^-)^2\{u_j L_j{}''(\alpha) + u_{j+1/2}L_{j+1/2}{}''(\alpha) + \bar{u}_\alpha L_\alpha{}''(\alpha)\} \qquad (2.16) \\ &\quad = (\beta^+)^2\{\bar{u}_\alpha H_{\alpha,0}{}''(\alpha) + \bar{u}'_\alpha H_{\alpha,1}{}''(\alpha) + u_{j+1}H_{j+1}{}''(\alpha)\}. \end{aligned}$$

This is a linear system for $\bar{u}_\alpha$ and $\bar{u}'_\alpha$ and the determinant of its coefficient matrix is $\Delta_5$. It can be easily shown that $\Delta_5 > 0$; hence this system must have a unique solution and the piecewise function defined by the second and the third formulas in (2.11) are uniquely determined by $u_j, u_{j+1/2}, u_{j+1}$ and the three jump conditions at $\alpha$, which must be the interpolant of $u$ on the interface element $e_j$. Therefore (2.11) is proven.

The rest results of this lemma follow from solving the above linear system for $\bar{u}_\alpha$ and then use (2.16) to represent $\bar{u}'_\alpha$. □

By the standard analysis, we know that

$$\left\|u - \tilde{I}_{h,\alpha}u\right\|_{0,\alpha} + h\left\|u - \tilde{I}_{h,\alpha}u\right\|_{1,\alpha} \le Ch^3 \|u'''\|_{0,\alpha}. \tag{2.17}$$

Hence, to estimate the error in $I_h u$, we can consider the difference between $I_h u$ and $\tilde{I}_{h,\alpha}u$ because $I_h u - u = I_h u - \tilde{I}_{h,\alpha}u + \tilde{I}_{h,\alpha}u - u$. Because of Lemma 2, we have

$$I_h u - \tilde{I}_{h,\alpha}u = \begin{cases} 0, & x \notin (x_j, x_{j+1}), \\ (\bar{u}_\alpha - u(\alpha))L_\alpha(x), & x \in (x_j, \alpha), \\ (\bar{u}_\alpha - u(\alpha))H_{\alpha,0}(x) & \\ \;+(\bar{u}'_\alpha - u'(\alpha^+))H_{\alpha,1}(x), & x \in (\alpha, x_{j+1}). \end{cases} \tag{2.18}$$

Hence we only need to estimate $I_h u - \tilde{I}_{h,\alpha}u$ on the interface element, and this can be done by studying $(\bar{u}_\alpha - u(\alpha))$ and $(\bar{u}'_\alpha - u'(\alpha^+))$.

By straightforward calculations, we can have

$$\begin{aligned} \bar{u}_\alpha - u(\alpha^-) &= \frac{1}{\Delta_5}\left\{\beta^-(\beta^+)^2 H''_{\alpha,1}(\alpha)e'(\alpha^-) - (\beta^-)^2\beta^+ e''(\alpha^-)\right. && (2.19)\\ &\quad \left.+(\beta^+)^3 e''(\alpha^+) - (\beta^-)^2\beta^+ u''(\alpha^-) + (\beta^+)^3 u''(\alpha^+)\right\} \\ \bar{u}'_\alpha - u'(\alpha+) &= \frac{1}{(\beta^+)^2 H''_{\alpha,1}(\alpha)}\left\{(\beta^-)^2 e''(\alpha^-) - (\beta^+)^2 e''(\alpha^+)\right., && (2.20)\\ &\quad +(\bar{u}_\alpha - u(\alpha-))((\beta^-)^2 L''_\alpha(\alpha) - (\beta^+)^2 H''_{\alpha,0}(\alpha)) \\ &\quad \left.+(\beta^-)^2 u''(\alpha^-) - (\beta^+)^2 u''(\alpha^+)\right\}, \end{aligned}$$

where $e(x) = \tilde{I}_{h,\alpha}u(x) - u(x)$. From the standard error estimates about quadratic interpolation we can see that

$$\begin{aligned} &|e'(\alpha^-)| \le (\alpha - x_{j+1/2})(\alpha - x_j)^{1/2}||u'''||_{L^2(x_j,\alpha)}, \\ &|e''(\alpha^-)| \le (\alpha - x_j)^{1/2}||u'''||_{L^2(x_j,\alpha)}, \\ &|e''(\alpha^+)| \le (x_{j+1} - \alpha)^{1/2}||u'''||_{L^2(\alpha,x_{j+1})}. \end{aligned} \tag{2.21}$$

Applying the estimates in (2.21) to $\bar{u}_\alpha - u(\alpha^-)$ we have

$$\begin{aligned} \left|\bar{u}_\alpha - u(\alpha^-)\right| &\le \frac{1}{\Delta_5}\left|\beta^-(\beta^+)^2 H''_{\alpha,1}(\alpha)e'(\alpha^-) - (\beta^-)^2\beta^+ e''(\alpha^-) + (\beta^+)^3 e''(\alpha^+)\right| \\ &\quad +\frac{1}{\Delta_5}\left\{(\beta^-)^2\beta^+\left|u''(\alpha^-)\right| + (\beta^+)^3\left|u''(\alpha^+)\right|\right\} \\ &\le Ch^{3/2}(\alpha - x_{j+1/2})\|u\|_{3,\alpha} + Ch(\alpha - x_{j+1/2})\|u\|_{3,\alpha}. \end{aligned} \tag{2.22}$$

Here we have used the facts that

$$\begin{aligned} \frac{1}{\Delta_5} &\le C(x_{j+1} - \alpha)(\alpha - x_{j+1/2}), \\ \left|u''(\alpha^\pm)\right| &\le C[\|u'''\|_{0,\alpha} + \|u''\|_{0,\alpha}] \le C||u||_{3,\alpha}. \end{aligned}$$

Apply (2.8) and (2.22) to the second formula in (2.18) we can have

$$||I_h u - \tilde{I}_{h,\alpha} u||_{L^2(x_j,\alpha)} + h||(I_h u - \tilde{I}_{h,\alpha} u)'||_{L^2(x_j,\alpha)} \leq Ch^{5/2}||u||_{3,\alpha}. \tag{2.23}$$

For the estimation on the interval $(\alpha, x_{j+1})$, we first note that

$$\begin{aligned}
& |\bar{u}_\alpha - u(\alpha)| \\
\leq\ & \frac{1}{\Delta_5}\left|\beta^-(\beta^+)^2 H''_{\alpha,1}(\alpha)e'(\alpha^-) - (\beta^-)^2\beta^+ e''(\alpha^-) + (\beta^+)^3 e''(\alpha^+)\right| \\
& + \frac{1}{\Delta_5}\left\{(\beta^-)^2\beta^+\left|u''(\alpha^-)\right| + (\beta^+)^3\left|u''(\alpha^+)\right|\right\} \\
\leq\ & Ch^{3/2}(x_{j+1}-\alpha)\left\|u'''\right\|_{0,\alpha} + Ch(x_{j+1}-\alpha)\left\|u\right\|_{3,\alpha},
\end{aligned} \tag{2.24}$$

here we have used the fact:

$$\frac{1}{\Delta_5} \leq C(x_{j+1}-\alpha)^2.$$

Also, using (2.20) and (2.22), we have

$$\begin{aligned}
& \left|\bar{u}'_\alpha - u'(\alpha^+)\right| \\
\leq\ & C(x_{j+1}-\alpha)(\alpha - x_j)^{1/2} + C(x_{j+1}-\alpha)(x_{j+1}-\alpha)^{1/2} \\
& + C\frac{(x_{j+1}-\alpha)\left|\bar{u}_\alpha - u(\alpha)\right|}{(\alpha - x_{j+1/2})(\alpha - x_j)} + C\frac{\left|\bar{u}_\alpha - u(\alpha)\right|}{x_{j+1}-\alpha} + C(x_{j+1}-\alpha)\left\|u\right\|_{3,\alpha} \\
\leq\ & C\left(h^{3/2} + h^{3/2}\frac{\alpha - x_{j+1/2}}{x_{j+1}-\alpha} + h\right)\left\|u\right\|_{3,\alpha},
\end{aligned} \tag{2.25}$$

here we have used the facts that

$$x_j < x_{j+1/2} < \alpha < x_{j+1},\ x_{j+1/2} = \frac{x_j + x_{j+1}}{2}.$$

Now using (2.25), (2.24), (2.9), and (2.10) in (2.18), we have

$$||I_h u - \tilde{I}_{h,\alpha} u||_{L^2(\alpha,x_{j+1})} + h||(I_h u - \tilde{I}_{h,\alpha} u)'||_{L^2(\alpha,x_{j+1})} \leq Ch^{5/2}||u||_{3,\alpha}. \tag{2.26}$$

Finally, using (2.23), (2.26), and (2.18), we can obtain an error estimate of $I_h u \in \tilde{S}_h(\Omega)$ in the following theorem which confirms our numerical experiments about the order of accuracy of interpolant formed in the hierarchical quadratic IFE space [8].

**Theorem 2.** *Assume that $u \in H^3_\alpha(0,1)$ and $I_h u \in \tilde{S}_h(\Omega)$ is its interpolation. Then there exits a positive constant $C > 0$, independent of $h$ and $u$, such that*

$$||I_h u - u||_{0,\alpha} + h||I_h u - u||_{1,\alpha} \leq Ch^{5/2}||u||_{3,\alpha}.$$

□

2.3. **A quadratic IFE space with an extra jump condition.** We now consider the quadratic IFE space with an extra second order derivative jump condition, see below and [8]. On each non-interface element $e_k$, this quadratic IFE space uses the standard local quadratic finite element space $S^2_h(e_k)$. On the interface element $e_j$, it uses local nodal basis functions $\bar{\psi}_{j,i}(x), i = 1,2,3$ defined as follows

- $\bar{\psi}_{j,i}|_{[x_j,\alpha]} \in P_2([x_j,\alpha]), \bar{\psi}_{j,i}|_{[\alpha,x_{j+1}]} \in P_2([\alpha,x_{j+1}])$ where $P_2$ is the set of polynomials of degree up to 2.
- $\bar{\psi}_{j,i}(t_{j,i}) = 1,\ \bar{\psi}_{j,l}(t_{j,l}) = 0,\ l \neq j.$
- $[\bar{\psi}_{j,i}]_{x=\alpha} = [\beta\bar{\psi}'_{j,i}]_{x=\alpha} = [\beta\bar{\psi}''_{j,i}]_{x=\alpha} = 0.$

We let

$$\bar{S}_h^2(e_j) = span\{\bar{\psi}_{j,1}, \bar{\psi}_{j,2}, \bar{\psi}_{j,3}\}.$$

We then define the quadratic IFE space $\bar{S}_h(\Omega)$ with an extra jump condition as follows: $v \in \bar{S}_h(\Omega)$ if

- $v \in C(\bar{\Omega})$.
- $v|_{e_k} \in S_h^2(e_k),\ k \neq j,\ v|_{e_j} \in \bar{S}_h^2(e_j)$.

Please refer to [8] for other properties of this quadratic IFE space.

**Lemma 3.** *The quadratic IFE interpolant $I_h u(x) \in \bar{S}_h(\Omega)$ has the same representation given in (2.11) with*

$$\begin{aligned} \bar{u}_\alpha &= \Delta_3^{-1}\{\beta^-\beta^+ H_{\alpha,1}{}''(\alpha)[u_j L_j'(\alpha) + u_{j+1/2}L_{j+1/2}'(\alpha)] \\ &\quad +\beta^+[\beta^+ u_{j+1}H''{}_{j+1}(\alpha) - \beta^-(u_j L''{}_j(\alpha) + u_{j+1/2}L''{}_{j+1/2}(\alpha))]\}, \end{aligned} \tag{2.27}$$

$$\begin{aligned} \bar{u}_\alpha' &= \frac{1}{\beta^+ H''{}_{\alpha,1}(\alpha)}\left\{\beta^-[u_j L''{}_j(\alpha) + u_{j+1/2}L''{}_{j+1/2}(\alpha) + \bar{u}_\alpha L''{}_\alpha(\alpha)]\right. \\ &\quad \left. -\beta^+[\bar{u}_\alpha H''{}_{\alpha,0}(\alpha) + u_{j+1}H''{}_{j+1}(\alpha)]\right\}, \end{aligned} \tag{2.28}$$

$$\Delta_3 = -\beta^-\beta^+ L_j'(\alpha)H''{}_{\alpha,1}(\alpha) + \beta^+[\beta^- L''{}_\alpha(\alpha) - \beta^+ H''{}_{\alpha,0}(\alpha)]. \tag{2.29}$$

*Proof.* The arguments are similar those used in Lemma 2 except that we use a different set of interface jump conditions: $[\beta(Iu)']_{x=\alpha} = 0,\ [\beta(I_h u)'']_{x=\alpha} = 0.$ ☐

We now assume that $u \in H_\alpha^3(\Omega)$ also satisfy $[\beta u'']_{x=\alpha} = 0$. Then, from (2.27), (2.28), and the interface jump conditions of $u$, we have

$$\bar{u}_\alpha - u_\alpha = \Delta_3^{-1}\left\{\beta^-\beta^+ H''{}_{\alpha,1}(\alpha)e'(\alpha^-) + (\beta^+)^2 e''(\alpha^+) - \beta^-\beta^+ e''(\alpha^-)\right\}, \tag{2.30}$$

and

$$\begin{aligned} \bar{u}_\alpha' - u'(\alpha^+) &= \frac{1}{\beta^+ H''{}_{\alpha,1}(\alpha)}\left\{\beta^- e''(\alpha^-) - \beta^+ e''(\alpha^+)\right. \\ &\quad \left. +(\bar{u}_\alpha - u_\alpha)(\beta^- L''{}_\alpha(\alpha) - \beta^+ H''{}_{\alpha,0}(\alpha))\right\}, \end{aligned} \tag{2.31}$$

where $e(x) = \tilde{I}_{h,\alpha}u(x) - u(x)$. Using the same arguments we can see that $|\bar{u}_\alpha - u_\alpha|$ and $|\bar{u}_\alpha' - u'(\alpha^+)|$ have similar estimates as (2.22), (2.24), and (2.25) except for the last term in each of them. Then, following arguments similar to those used for (2.23) and (2.26), we have

$$||I_h u - \tilde{I}_{h,\alpha}u||_{L^2(x_j,\alpha)} + h||(I_h u - \tilde{I}_{h,\alpha}u)'||_{L^2(x_j,\alpha)} \le Ch^3||u'''||_{L^2(x_j,x_{j+1})}. \tag{2.32}$$

and

$$||I_h u - \tilde{I}_{h,\alpha}u||_{L^2(\alpha,x_{j+1})} + h||(I_h u - \tilde{I}_{h,\alpha}u)'||_{L^2(\alpha,x_{j+1})} \le Ch^3||u'''||_{L^2(x_j,x_{j+1})}. \tag{2.33}$$

Finally, applying (2.17), (2.32), and (2.33) to $I_h u - u = I_h u - \tilde{I}_h u + \tilde{I}_h u - u$, we can obtain the error estimate for $I_h u(x)$ as stated in the following theorem.

**Theorem 3.** *Assume that $u \in H_\alpha^3(0,1)$ is such that $[\beta u''] = 0$ and $I_h u \in \bar{S}_h(\Omega)$ is its interpolation. Then there exits a positive constant $C > 0$, independent of $h$ and $u$, such that*

$$||I_h u - u||_{0,\alpha} + h||I_h u - u||_{1,\alpha} \le Ch^3||u||_{3,\alpha}.$$

☐

3. **Applications of the IFE spaces and imperfect contact interfaces.** In this section, we consider applications of the IFE spaces discussed in the previous section to interface problems. Our goal is to demonstrate that the related IFE solutions have the same orders of accuracy as their interpolation counter parts. Another topic is to extend the IFE methods to other types of interface problems. As specific examples, we discuss interface problems with non-homogeneous jump conditions and interface problems with imperfect interface jump conditions.

3.1. **The accuracy of the IFE solutions to interface problems.** We first consider interface (1.2)-(1.3) with homogeneous boundary and interface jump conditions. The weak form of this interface problem is: find $u(x) \in H^1_{0,\alpha}(\Omega)$ such that

$$A_\alpha(u,v) = (f,v), \qquad v \in H^1_{0,\alpha}, \tag{3.1}$$

where $H^1_{0,\alpha}(\Omega)$ is the subspace of $H^1_\alpha(\Omega)$ whose elements have zero values on the boundary of $\Omega$, and

$$A_\alpha(u,v) = \int_0^\alpha \beta^-(x)u'v'dx + \int_\alpha^1 \beta^+(x)u'v'dx, \tag{3.2}$$

and $(\cdot,\cdot)$ is the standard $L^2$ inner product.

Let $V_h$ be the space formed by functions with zero boundary values from one of the IFE spaces discussed in the previous section. The IFE solution to the interface problem generated from this IFE space is the function $u_h \in V_h$ such that

$$A_\alpha(u_h,v_h) = (f,v_h), \qquad v_h \in V_h. \tag{3.3}$$

First we note that these IFE methods are conforming methods because $V_h \subset H^1_{0,\alpha}(\Omega)$. Secondly, the bilinear form $A_\alpha(\cdot,\cdot)$ is obviously symmetric, bounded, and coercive. Then, by the usual finite element error estimation procedure for linear elliptic boundary value problems and the error estimates obtained for the interpolants in the previous section, we can easily obtain the error estimates for the IFE solutions stated in the following theorem.

**Theorem 4.** *Assuming that the exact solution $u(x)$ to the interface problem has the required regularity implied in the estimates below, then the IFE solutions to the interface problem have the following error estimates:*

$$\|u-u_h\|_0 + h\,\|u-u_h\|_{1,\alpha} \le \begin{cases} Ch^2\,\|u\|_{2,\alpha}, & \textit{when } V_h = H^1_{0,\alpha}\cap S^1_h(\Omega), \\ Ch^{5/2}\,\|u\|_{3,\alpha}, & \textit{when } V_h = H^1_{0,\alpha}\cap \tilde{S}^2_h(\Omega), \\ Ch^3\,\|u\|_{3,\alpha}, & \textit{when } V_h = H^1_{0,\alpha}\cap \bar{S}^2_h(\Omega). \end{cases}$$

□

The IFE spaces discussed in this paper can also be used to handle interface problems whose jump conditions are not homogeneous even though functions in these IFE spaces are constructed according to the homogeneous jump conditions. This can be achieved through the usual homogenization procedure. For example, let us consider the general interface problem (1.2)-(1.3) with nonhomogeneous boundary and interface jump conditions. We can first construct a piecewise smooth function $\psi$ as follows

$$[\psi]_{x=\alpha} = A, \qquad [\beta\psi']_{x=\alpha} = B,\ \psi(0)=u_0,\ \psi(1)=u_1. \tag{3.4}$$

Now letting $U = u - \psi$, we can see that $U$ is the solution to the following interface problem with homogeneous boundary and interface jump conditions:

$$-(\beta(x)U')' = F, \; x \in (0,1), \tag{3.5}$$

$$[U] = [\beta U'] = 0 \quad \text{at} \quad x = \alpha, \qquad U(0) = U(1) = 0. \tag{3.6}$$

where $F = f + \beta'\psi'$ is a piecewisely defined on $(0,\alpha)$ and $(\alpha,1)$.

Thus by letting $U_h$ be an IFE solution of (3.5 -3.6) we can obtain the IFE solution $u_h = U_h + \psi$ to the interface problem (1.2)-(1.3) with the accuracy stated in Theorem 4.

For quadratic element case studied in §2.3, a piecewise quadratic function $\psi$ can be constructed by

$$[\psi]_{x=\alpha} = A, \quad [\beta\psi']_{x=\alpha} = B, \quad [\beta\psi'']_{x=\alpha} = [f]_{x=\alpha}, \qquad \psi(0) = u_0, \; \psi(1) = u_1.$$

so that $u = U + \psi$ with $U$ satisfying the homogeneous jump interface conditions. Thus the IFE can be applied to compute numerical solution $U_h$ of $U$ (which satisfies a corresponding different equation with homogeneous jump conditions), then IFE solution $u_h$ is defined by $u_h = U_h + \psi$.

Also, we would like to point out all the discussions and results above can be readily extended to the case in which the boundary problem has finite many interfaces.

3.2. **Imperfect contact interface problems using linear elements.** In this section we consider using IFE spaces to solve an interface problem with "imperfect" contact conditions at the interface, see [14] and references therein for more details about this type of interface problems.

Specifically, we consider the problem in which we want to find a function $u$ such that

$$-(\beta u')' = f, \qquad x \in (0,1), \tag{3.7}$$

$$[u]_{x=\alpha} = \lambda\beta^- u'(\alpha^-) = \lambda\beta^+ u'(\alpha^+), \qquad u(0) = u(1) = 0, \tag{3.8}$$

where $\lambda > 0$, $\beta = \beta^-$ for $x \in (0,\alpha)$ and $\beta = \beta^+$ for $x \in (\alpha,1)$ are positive constants, and $f \in L^2(0,1)$. In fact $\lambda > 0$ is a physical constant related to the gap of the imperfect contact of two materials. The main difference between the standard interface jump condition and the imperfect jump condition (3.8) is that the jump in $u$ is unknown a-priori at $x = \alpha$.

Let $H^{1,\lambda}_\alpha(0,1)$ be space defined by

$$H^{1,\lambda}_\alpha(0,1) = \left\{ u \in L^2(0,1) \mid u \in H^1(0,\alpha) \cap H^1(\alpha,1), \right.$$
$$\left. [u]_{x=\alpha} = \lambda\beta^- u'(a^-) = \lambda\beta^+ u'(\alpha^+) \right\}.$$

As usual, we use $H^{1,\lambda}_{0,\alpha}(0,1)$ to denote the space formed by functions from $H^{1,\lambda}_\alpha(0,1)$ whose values on the boundary are zero.

First, we introduce a bilinear form related with our imperfect interface problem: for $u, \; v \in H^{1,\lambda}_\alpha(0,1)$ we let

$$A_{\alpha,\lambda}(u,v) = \left.\frac{[u]\,[v]}{\lambda}\right|_{x=\alpha} + \int_0^\alpha \beta^-(x)u'v'dx + \int_\alpha^1 \beta^+(x)u'v'dx. \tag{3.9}$$

Then the weak form of (3.7)-(3.8) is to find $u \in H^{1,\lambda}_{0,\alpha}(0,1)$ such that

$$A_{\alpha,\lambda}(u,v) = (f,v), \qquad \text{for any } \; v \in H^{1,\lambda}_{0,\alpha}(0,1). \tag{3.10}$$

Simple calculations can show that this bilinear form has the usual coercivity and boundedness as sated in the following theorem.

**Theorem 5.** *Assume that $f \in L^2(0,1)$, then the weak problem (3.10) has a unique solution that satisfies*

$$C^{-1}||u||_{1,\alpha} \le A_{\alpha,\lambda}(u,u) \le C||f||,$$

*for a positive constant $C > 0$.*

□

All the IFE spaces discussed in the previous section can be extended to handle this imperfect interface problem. Without loss of generality, we present details about extending the linear IFE space. As before, at the node $x_j$, we define a piecewise linear function $\phi_j^\lambda(x)$ such that

$$\phi_j^\lambda(x_i) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \ne j, \end{cases}$$
$$[\phi_j^\lambda]_{x=\alpha} = \lambda\beta^-(\phi_j^\lambda)'(\alpha^-) = \lambda\beta^+(\phi_j^\lambda)'(\alpha^+).$$

Then we define a linear IFE space for the interface problem (3.7)-(3.8) by

$$S_h^{1,\lambda}(\Omega) = span\{\phi_j^\lambda, j = 0, 1, \cdots, N\}.$$

Now, for any $u \in H_\alpha^{1,\lambda}(0,1)$, we let $I_h u(x) \in S_h^{1,\lambda}(\Omega)$ be its interpolation, and let $\tilde{I}_{h,\alpha}u$ be the standard piecewise linear Lagrange interpolation of $u$ defined on the partition formed by $x_j, j = 0, 1, \cdots, N$ and $\alpha$.

Again, we assume that $e_j$ is the only interface element. On this element, there are two constants $u_I^-$ and $u_I^+$ such that

$$I_h u(x) = \begin{cases} u_j L_{j,0}(x) + u_I^- L_{j,\alpha}(x), & x \in (x_j, \alpha), \\ u_I^+ L_{j+1,\alpha}(x) + u_{j+1} L_{j+1,1}(x), & x \in (\alpha, x_{j+1}). \end{cases}$$

In fact, it follows from the jump interface condition that $u_I^-$ and $u_I^+$ satisfy

$$\begin{aligned} u_I^+ - u_I^- &= \lambda\beta^-(I_h u)'(\alpha^-) = \lambda\beta^-(u_j L'_{j,0} + u_I^- L'_{j,\alpha}), \\ u_I^+ - u_I^- &= \lambda\beta^+(I_h u)'(\alpha^+) = \lambda\beta^+(u_I^+ L'_{j+1,\alpha} + u_{j+1} L'_{j+1,1}). \end{aligned}$$

Solving this linear system for $u_I^-$ and $u_I^+$ we have

$$u_I^- = \frac{-\lambda\beta^- u_j L'_{j,0}(1 - \lambda\beta^+ L'_{j+1,\alpha}) + \lambda\beta^+ u_{j+1} L'_{j+1,1}}{\Delta_1}, \tag{3.11}$$

$$u_I^+ = \frac{\lambda\beta^+ u_{j+1} L'_{j+1,1}(1 + \lambda\beta^- L'_{j,\alpha}) - \lambda\beta^- u_j L'_{j,0}}{\Delta_1}, \tag{3.12}$$

$$\Delta_1 = -\lambda\beta^+ L'_{j+1,\alpha}(1 + \lambda\beta^- L'_{j,\alpha}) + \lambda\beta^- L'_{j,\alpha}. \tag{3.13}$$

On the other hand, we have

$$\tilde{I}_{h,\alpha}u(x) = \begin{cases} u_j L_{j,0}(x) + u(\alpha^-) L_{j,\alpha}(x), & x \in (x_j, \alpha), \\ u(\alpha^+) L_{j+1,\alpha}(x) + u_{j+1} L_{j+1,1}(x), & x \in (\alpha, x_{j+1}). \end{cases}$$

Hence

$$I_h u - \tilde{I}_{h,\alpha}u = \begin{cases} 0, & x \notin e_j = (x_j, x_{j+1}), \\ (u_I^- - u(\alpha^-)) L_{j,\alpha}(x), & x \in (x_j, \alpha), \\ (u_I^+ - u(\alpha^+)) L_{j+1,\alpha}(x), & x \in (\alpha, x_{j+1}), \end{cases} \tag{3.14}$$

and it is therefore important to estimate the differences of $u_I^- - u(\alpha^-)$ and $u_I^+ - u(\alpha^+)$.

By a simple calculations and the jump conditions satisfied by $u$, we can see that

$$u_I^+ - u(\alpha^+) = \frac{J_1 + J_2 + J_3}{\Delta_1}, \tag{3.15}$$

where

$$J_1 = -\lambda\beta^- e'(\alpha^-),\ J_2 = \lambda\beta^+ e'(\alpha^+),\ J_3 = \lambda^2\beta^-\beta^+ e'(\alpha^+))L'_{j,\alpha}$$

with $e = \tilde{I}_{h,\alpha}u - u$. Because $e(x_j) = e(\alpha^-) = e(\alpha^+) = e(x_{j+1}) = 0$, we can follow the standard procedure to obtain

$$\begin{aligned} |J_1| &\leq \lambda\beta^-(\alpha - x_j)^{1/2}\left\|u''\right\|_{L^2((x_j,\alpha))}, \\ |J_2| &\leq \lambda\beta^+(x_{j+1} - \alpha)^{1/2}\left\|u''\right\|_{L^2((\alpha,x_{j+1}))}, \\ |J_3| &\leq \lambda^2\beta^-\beta^+(x_{j+1} - \alpha)^{1/2}(\alpha - x_j)^{-1}\left\|u''\right\|_{L^2((\alpha,x_{j+1}))}. \end{aligned}$$

Hence, it follows from (3.15) that

$$|u_I^+ - u(\alpha^+)| \leq Ch^{1/2}(x_{j+1} - \alpha)(\|u''\|_{L^2((x_j,\alpha))} + \|u''\|_{L^2((\alpha,x_{j+1})}),$$

which leads to

$$\begin{aligned} & ||I_h u - \tilde{I}_{h,\alpha}u||_{L^2((\alpha,x_{j+1}))} + h||(I_h u - \tilde{I}_{h,\alpha}u)'||_{L^2([\alpha,x_{j+1}])} \\ \leq\ & Ch^2\left(||u''||_{L^2(x_j,\alpha)} + ||u''||_{L^2(\alpha,x_{j+1})}\right), \end{aligned}$$

Applying a similar estimate to $u_I^- - u(\alpha^-)$ we can obtain

$$\begin{aligned} & ||I_h u - \tilde{I}_{h,\alpha}u||_{L^2(x_j,\alpha)} + h||(I_h u - \tilde{I}_{h,\alpha}u)'||_{L^2(x_j,\alpha)} \\ \leq\ & Ch^2\left(||u''||_{L^2(x_j,\alpha)} + ||u''||_{L^2(\alpha,x_{j+1})}\right). \end{aligned}$$

We can now derive an error estimate for $I_h u(x)$ in the following theorem.

**Theorem 6.** *Let $u \in H^{1,\lambda}_{0,\alpha}(0,1)$ be such that $u|_{(0,\alpha)} \in H^2(0,\alpha)$ and $u|_{(\alpha,1)} \in H^2(\alpha,1)$. Then there exits a positive constant $C > 0$, independent of $u$ and $h$, such that*

$$||I_h u - u||_{0,\alpha} + h||I_h u - u||_{1,\alpha} \leq Ch^2||u||_{2,\alpha}.$$

*Proof.* The result can be obtained by applying the estimate above to (3.14), error estimates for the standard linear interpolation, and triangle inequality. □

Now, we consider the IFE solution $u_h \in S_h^{1,\lambda}(\Omega) \cap H^{1,\lambda}_{0,\alpha}(0,1)$ for the interface problem (3.7)-(3.8) defined by

$$A_{\alpha,\lambda}(u_h, v_h) = (f, v_h), \quad \text{for any } \ v_h \in S_h^\lambda \cap H^{1,\lambda}_{0,\alpha}(0,1). \tag{3.16}$$

**Theorem 7.** *Assume that the weak solution $u$ of (3.7)-(3.8) is such that $u|_{(0,\alpha)} \in H^2(0,\alpha)$ and $u|_{(\alpha,1)} \in H^2(\alpha,1)$. Then there exits a positive constant $C > 0$, independent of $u$ and $h$, such that*

$$||u - u_h||_{0,\alpha} + h||u - u_h||_{1,\alpha} \leq Ch^2||u||_{2,\alpha}.$$

*Proof.* The estimate can be derived from Theorems 5, Theorem 6, and the routine procedure for finite element error estimation of elliptic problems. □

**Remark 3.1.** As before, following a similar homogenization procedure, we can use the IFE space above to solve those interface problems with non-homogeneous imperfect contact interface conditions, i.e.,

$$[u]_{x=\alpha} - \lambda b^- u'(\alpha^-) = A, \qquad [u]_{x=\alpha} - \lambda b^+ u'(\alpha^+) = B,\ u(0) = u_0,\ u(1) = u_1.$$

3.3. **Imperfect contact interface problems using quadratic elements.** It is easy to see that the hierarchical quadratic elements are not applicable here due to the nature of the contact interfaces, thus we only consider the quadratic element with an extra jump condition at the second order derivative $[\beta u'']_{x=\alpha} = 0$. Following §2.3 and §3.2 above, The quadratic IFE interpolant $I_h u(x) \in S_h^{2,\lambda}$ is defined by

$$I_h u(x) = \begin{cases} \tilde{I}_{h,\alpha} u(x), & x \notin (x_j, x_{j+1}), \\ u_j L_j(x) + u_{j+1/2} L_{j+1/2}(x) + u_I^- L_\alpha(x), & x \in (x_j, \alpha), \\ u_I^+ H_{\alpha,0}(x) + \bar{u}'_\alpha H_{\alpha,1}(x) + u_{j+1} H_{j+1}(x), & x \in (\alpha, x_{j+1}), \end{cases} \tag{3.17}$$

where $S_h^{2,\lambda}$ is the quadratic IFE element spaces with imperfect contact interface conditions (3.8) and the extra second order derivative jump condition. It follows from the jump conditions (3.8) and $[\beta (I_h u)'']_{x=\alpha} = 0$ that

$$\begin{aligned} &u_I^+ - u_I^- = \lambda \beta^- \{u_j L'_j(\alpha) + u_{j+1/2} L'_{j+1/2}(\alpha) + u_I^+ L'_\alpha(\alpha)\}, \\ &u_I^+ - u_I^- = \lambda \beta^+ \bar{u}'_\alpha, \\ &\beta^- [u_j L''_j(\alpha) + u_{j+1/2} L''_{j+1/2}(\alpha) + u_I^+ L''_\alpha(\alpha)] \\ &\qquad\qquad = \beta^+ [u_I^+ H''_{\alpha,0}(\alpha) + \bar{u}'_\alpha H''_{\alpha,1}(\alpha) + u_{j+1} H''_{j+1}(\alpha)]. \end{aligned}$$

The above is a linear systems for $u_I^+$, $u_I^-$ and $\bar{u}'_\alpha$, and the determinant of this system is given by

$$\Delta_0 = -\lambda \beta^- L''_\alpha [\beta^+ H''_{\alpha,1} + \lambda (\beta^+)^2 H''_{\alpha,0}] - \lambda \beta^+ [\beta^+ H''_{\alpha,0} - \beta^- L''_\alpha] > 0,$$

which is positive for all $\lambda, \beta >$ and $x_j < x_{j+1/2} < \alpha < x_{j+1}$. Thus the interpolation operator $I_h u$ is well-defined. The analysis on the error estimates for the interpolations and IFE solution can be carried out in a similar fashion presented in the previous sections, we therefore omit the proofs of the following results.

**Theorem 8.** *(I) Let $u \in H_{0,\alpha}^{1,\lambda}(0,1)$ be such that $u \in H_{0,\alpha}^3(0,1)$. Then there exits a positive constant $C > 0$, independent of $u$ and $h$, such that*

$$||I_h u - u||_{0,\alpha} + h||I_h u - u||_{1,\alpha} \le Ch^3 ||u||_{3,\alpha}.$$

*(II) Assume that the weak solution $u$ of (3.7)-(3.8) is such that $u \in H_{0,\alpha}^3(0,1)$ . Then there exits a positive constant $C > 0$, independent of $u$ and $h$, such that the IFE solution $u_h \in S_h^{2,\lambda}$ of (3.16) (with $S_h^{1,\lambda}$ replaced by $S_h^{2,\lambda}$) satisfies*

$$||u - u_h||_{0,\alpha} + h||u - u_h||_{1,\alpha} \le Ch^3 ||u||_{3,\alpha}.$$

□

**Remark 3.2.** It is clearly seen that the homogenization procedure similar to that described in §3.1 can also be extended to the quadratic case when the second order jump is not zero.

**Acknowledgements.** This research was partially supported by the NSF grant DMS-9704621, DMS-0073403, the Army grant ARO-39676-MA, NSERC (Canada) and the Research Grants Council of the Hong Kong Special Administration Region, China (Project No. CityU 102005).

## REFERENCES

[1] S. Ajerid and T. Lin, *Higher-order immersed discontinuous galerkin methods*, International Journal of Information & Systems Sciences, Vol 3, No 4 (2007), 556-569

[2] I. Babuška, *The finite element method for elliptic equations with discontinuous coefficients*, Computing, **5** (1970), 207–213.

[3] I. Babuška and J.E. Osborn, *Generalized finite element methods: their performance and relation to mixed methods*, SIAM J. Numer. Anal., **20** (1983), 510–536.

[4] I. Babuška and J.E. Osborn, *Finite element methods for the solution of problems with rough input data*, in , "Singular and Constructive Methods for their Treatment, Lecture Notes in Mathematics" (**1121**) (eds. P. Grisvard, W. Wendland, and J.R. Whiteman), pages 1–18, New York, Springer-Verlag, 1985.

[5] Eric Bakker and A.J. Meir, *How do pulsed amperometric ion sensors work? A simple PDE miodel*, SIAM Review. **45** (2003), 327-344.

[6] J.B. Bell and D.L. Marcus, *A second-order projection method for variable-density flows*, J. Comput. Phys., **101** (1992), 334–348.

[7] J. Bramble and J. King, *A robust finite element method for nonhomogeneous Dirichlet problems in domains with curved boundaries*, Math. Comp., **63** (1994), 1–17.

[8] B. Camp, T. Lin, Y. Lin and W. Sun, *Quadratic immersed finite element spaces and their approximation capabilities*, Adv. Comput. Math., **24** (2006), 81–112.

[9] K. Chan, K. Zhang and Jun Zou, *Spherical interface dynamos: Mathematical theory, finite element approximations and application*, SIAM Numer. Anal., **44** (2006), 1877-1902.

[10] Z. Chen and J. Zou, *Finite element methods and their convergence for elliptic and parabolic interface problems*, Numer. Math., **79** (1998), 175–202.

[11] A. J. Chorin, *Numerical solution of the Navier-Stokes equations*, Math. Comp., **22** (1968), 745–762.

[12] E. Chung, Q. Du and J. Zou, *Convergence analysis of a finite volume method for Maxwell's equations in nonhomogeneous media*, SIAM J. Numer. Anal., **41** (2003), 37–63.

[13] A. Dadone and B. Grossman, *Progressive optimization of inverse fluid dynamic design problems*, Computers & Fluids, **29** (2000), 1–32.

[14] R. Ewing, OIliev and R. Lazarov, *A modified finite volume approximation of second order elliptic equations with discontinuous coefficients*, SIAM Sci. Comput., **23** (2001), 1335-1351.

[15] J. Haslinger and P. Neittaamäki, "Finite Element Approximation for Optimal Shape, Material and Topology Design," John Wiley & Sons, Ltd, Chichester, 1996.

[16] B. Heinrich, "Finite Difference Methods on Irregular Networks," volume 82 of International Series of Numerical Mathematics, Birkhäuser, Boston, 1987.

[17] G. Hetzer and A.J. Meir, *On an interface problem with a nonlinear jump condition, numerical approximation of solution*, International Jounral of Numerical Analysis and Modeling, **4** (2007), 519-530.

[18] D.W. Hewett, *The embedded curved boundary method for orthogonal simulation meshes*, J. Comput. Phys., **138** (1997), 585–616.

[19] J. Huang and J. Zou, *Some new a priori estimates for second-order elliptic and parabolic interface problems*, J. Differential Equations, **184** (2002), 570–586.

[20] J. Huang and J. Zou, *Uniform a priori estimates for elliptic and static maxwell interface problems*, Discrete and Continuous Dynamcial Systems, Series B, **7** (2006), 145-170.

[21] R. Kafafy, T. Lin, Y. Lin and T. Wang, *Three-dimensional immersed finite element methods for electric field simulation in composite materials*, International Journal of Numerical Methods in Engineering, **64** (2005), 940–972.

[22] J. Li, Y. Renardy and M. Renardy, *Numerical simulation of breakup of a viscous drop in simple shear flow through a volume-of-fluid method*, Phys. Fluids, **12** (2000), 269–282.

[23] Z. Li, *The immersed interface method using a finite element formulation*, Applied Numer. Math., **27** (1998), 253–267.

[24] Z. Li, T. Lin, Y. Lin and R. Rogers, *An immersed finite element space and its approximation capability*, Numerical Methods for Partial Differential Equations, **20** (2004), 338-367.

[25] Z. Li, T. Lin and X. Wu, *New cartesian grid methods for interface problems using finite element formulation*, Numerische Mathematik, **96** (2003), 61–98.

[26] T. Lin, Y. Lin, R.C. Rogers and L.M. Ryan, *A rectangular immersed finite element method for interface problems*, in "Advances in Computation: Theory and Practice" (eds. P. Minev and Y. Lin) , Vol. **7**, pages 107–114. Nova Science Publishers, Inc., 2001.

[27] T. Lin and J. Wang, *An immersed finite element electric field solver for ion optics modeling*, in "Proceedings of AIAA Joint Propulsion Conference, Indianapolis, IN, July, 2002," AIAA, 2002-4263.

[28] T. Lin and J. Wang, *The immersed finite element method for plasma particle simulation*, in "Proceedings of AIAA Aerospace Sciences Meeting, Reno, NV, Jan., 2003," AIAA, 2003-0842.

[29] M.P. Robinson and G. Fairweather, *Orthogonal spline collocation methods for schrödinger-type equations in one space variable*, Numer. Math., 68 (1994), 355–376.

[30] A.A. Samarskiǐ and V.B. Andreev, "Méthods aux Différences pour Équations Ellipticques," Mir, Moscow, 1978.

[31] Hongyun Wang, *An improved WPE method for solving discontinuous Fokker-Planck equations*, International Jounral of Numerical Analysis and Modeling, to appear in 2008.

[32] J. Wang, J. Polk, J. Brophy and I. Katz, *3-d particle simulations of nstar ion optics plasma flow*, IEPC 2001-085, (2001).

[33] S. Zhu, G. Yuan and W. Sun, *Convergence and stability of explicit/implicit schemes for parabolic equations with discontinuous coefficients*, International Journal of Numerical Analysis and Modelling, **1** (2004), 131-145.

Received October 2006; revised February 2007.

*E-mail address*: lin@math.vt.edu
*E-mail address*: ylin@math.ualberta.ca
*E-mail address*: mawsun@math.cityu.edu.hk