An Immersed Finite Element Space and Its Approximation Capability

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This article discusses an immersed finite element (IFE) space introduced for solving a second-order elliptic boundary value problem with discontinuous coefficients (interface problem). The IFE space is nonconforming and its partition can be independent of the interface. The error estimates for the interpolation of a function in the usual Sobolev space indicate that this IFE space has an approximation capability similar to that of the standard conforming linear finite element space based on body-fit partitions. Numerical examples of the related finite element method based on this IFE space are provided. © 2004 Wiley Periodicals, Inc. Numer Methods Partial Differential Eq 20: 338–367, 2004

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1. INTRODUCTION

In this article, we discuss the approximation capability of an immersed finite element (IFE) space formed by first degree polynomials for the following interface problem:

$$-\nabla \cdot (\beta \nabla u) = f, \qquad (x, y) \in \Omega, \tag{1.1}$$

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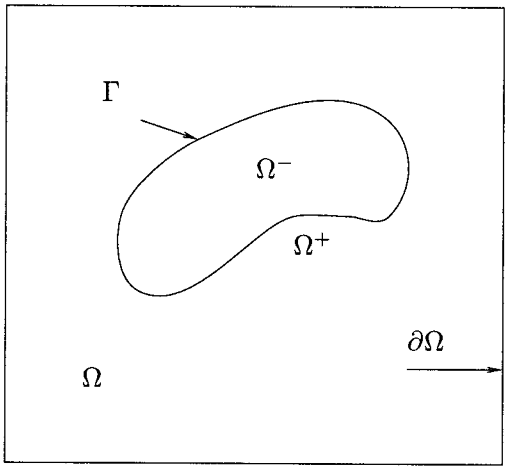

FIG. 1. A sketch of the domain for the interface problem.

$$u|_{\partial\Omega} = g, \tag{1.2}$$

together with the jump conditions on the interface Γ:

$$[u]|_\Gamma = 0, \tag{1.3}$$

$$[\beta u_n]|_\Gamma = 0. \tag{1.4}$$

Here, see the sketch in Fig. 1, $\Omega \subset \mathbb{R}^2$ is a convex polygonal domain, the interface Γ is a curve separating Ω into two subdomains Ω^-, Ω^+ such that $\Omega = \Omega^- \cup \Omega^+ \cup \Gamma$, and the coefficient $\beta(x, y)$ is a piecewise constant function defined by

$$\beta(x, y) = \begin{cases} \beta^-, & (x, y) \in \Omega^-, \\ \beta^+, & (x, y) \in \Omega^+. \end{cases}$$

This IFE space was introduced in [1], which reported some preliminary analysis and numerical results. The interface problem considered here appears in many engineering and science applications; see the related references in [1] and [2, 3]. This IFE space can also be used to handle interface problems with nonhomogeneous interface jump conditions [with a nonzero constant value on the right hand of (1.3) and/or (1.4)] by either simply modifying the IFE space [see the Remark after (2.5)], or reducing the interface problem to that with homogeneous interface jump conditions via the usual homogenization technique based on a change of variable. How to extend this IFE space for handling other cases, such as more general nonhomogeneous interface jump conditions and piecewise smooth instead of piecewise constant coefficient function β, leads to interesting future research projects.

It is well known [see [4, 5] and the references therein] that the standard Galerkin method with linear finite elements can be used to solve such elliptic interface problems. However, to achieve the optimal $O(h^2)$ accuracy in the numerical solutions, triangles are required to be aligned with the interface, i.e., the interface is allowed to pass a triangle only through its vertices. This restriction will obviously prevent the Galerkin method with linear finite elements from working efficiently for those applications in which an interface problem similar to the one defined by (1.1)–(1.4) has to be solved repeatedly, each time with a different interface Γ, because the partition has to be formed over and over again because of the variation (either the shape or the position) of Γ. In addition, many applications in which structured (such as Cartesian) partitions are preferred may also prevent the use of the standard finite element method if the involved problems have nontrivial interfaces.

The basic idea of the immersed finite elements is to form the partition $\mathcal{T}_h$ independent of interface Γ so that partitions with simple and efficient structures, such as a Cartesian partition, can be used to solve an interface problem with a rather complicated or varying interface. We only consider partitions formed by triangles here, partitions formed by quadrilateral elements were discussed in [6]. Without loss of generality, we assume that the triangles in the partition have the following features:

(H_1): If Γ meets one edge of a triangle at more than two points, then this edge is part of Γ.

(H_2): If Γ meets a triangle at two points, then these two points must be on different edges of this triangle.

Obviously, triangles in a partition can be separated into two classes:

- Noninterface triangle: The interface Γ either does not intersect with this triangle, or it intersects with this triangle but does not separate its interior into two nontrivial subsets.
- Interface triangle: The interface Γ cuts through its interior.

In a noninterface triangle, we use the standard linear polynomials as local nodal basis functions. However, in an interface triangle, we use piecewise linear polynomials defined in the two subsets formed by the interface in such a way that the functions satisfy the jump conditions (either exactly or approximately) on the interface and retain specified values at the vertices of the interface triangle. The idea here is similar to that used for the Hsieh-Clough-Tocher macro C^1 element [7] where each basis function consists of three cubic polynomials on the subtriangles formed by connecting the vertices and the center of gravity so that the required continuity can be satisfied. The immersed finite element space defined over the whole domain Ω with a partition chosen can then be constructed through the standard procedure. We refer the readers to [1, 8–17] for more background materials about immersed interface and immersed finite element methods as well as their applications. The main effort of this article is to investigate the approximation capability of the IFE space introduced in [1] to treat the interface problems, which is a critical step towards analyzing errors of a finite element (volume) solution to an interface problem based on this IFE space. Numerical examples generated by the finite element method based on this IFE space are also provided, but the related error estimation will be given in a forthcoming article.

This article is organized as follows. In Section 2, we introduce the IFE space and describe basic properties of its local nodal basis functions. In Section 3, we use the technique based on the multi-point Taylor expansion, see [18, 19], to derive error estimates for the interpolation in the IFE space of the functions in Sobolev spaces. Several arguments used in the error estimation here are inspired by [5]. In Section 4, we present several numerical examples generated by the related IFE method.

Here are some conventions used in this article. For any subset T of Ω, we let

$$T^s = T \cap \Omega^s, \qquad s = -, +.$$

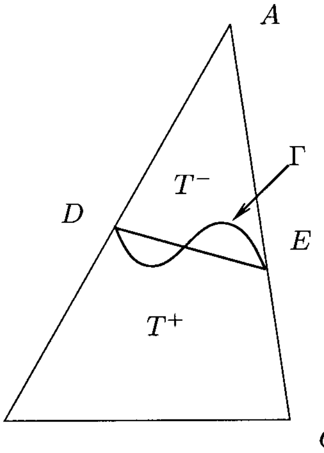

FIG. 2. A typical interface triangle ΔABC. The curve between D and E is part of the interface Γ.

For any function $f(x, y)$ defined in $T \subset \Omega$, we can restrict it to T^s, $s = -, +$ to obtain two functions as

$$f^s(x, y) = f(x, y), \qquad \text{if } (x, y) \in T^s, s = -, +.$$

We use $\overline{DE}$ to denote the line segment between two points $D, E \in \Omega$. For any curve Γ, we use $\mathbf{n}_\Gamma$ to denote its unit normal vector pointing to a particular side of Γ. For any measurable subset Λ of Ω, we use $|\Lambda|$ to denote its measure. In deriving estimates, we often use C to represent a generic constant whose value might be different from line to line. Also, in the discussion below we add assumptions as we progress, and all the assumptions made before any theorem or lemma are assumed to hold for that statement.

2. THE IMMERSED FINITE ELEMENT SPACE

In this section, we first introduce the local nodal IFE basis functions and then use them to define the IFE space over the whole domain with a partition chosen. We will also describe basic features of these basis functions.

For a typical triangle $T \in \mathcal{T}_h$, where $\mathcal{T}_h$ is a typical family of partitions of domain Ω, we use $A = (x_1, y_1)^T$, $B = (x_2, y_2)^T$, $C = (x_3, y_3)^T$ to denote its vertices, and uses $D = (x_D, y_D)^T$ and $E = (x_E, y_E)^T$ to denote its interface points on the edges if T is an interface triangle (see the sketch in Fig. 2.)

Our main concern is the finite element functions in an interface triangle $T \in \mathcal{T}_h$. We follow an idea similar to that for the Hsieh-Clough-Tocher macro C^1 element [7] in which piecewise polynomials are used in a triangle to maintain certain desirable features. For our interface problem, we obviously would like the finite element functions to satisfy the jump conditions across the interface. Because the interface Γ separates an interface triangle T into two subsets T^- and T^+, we naturally can try to form a finite element function piecewise by two first-degree polynomials defined in T^- and T^+, respectively. Note that each polynomial of degree one has three freedoms (coefficients). The values of the finite element function at the vertices of T

provide three restrictions. The normal derivative jump condition on $\overline{DE}$ provides another. Then we can have two more restrictions by requiring the continuity of the finite element function at interface points D and E. Intuitively, these six conditions can yield the desired piecewise linear polynomial in an interface triangle. This idea leads us to consider functions defined as follows:

$$\phi(x, y) = \begin{cases} \phi^-(x, y) = a_1 x + b_1 y + c_1, & (x, y) \in T^-, \\ \phi^+(x, y) = a_2 x + b_2 y + c_2, & (x, y) \in T^+, \\ \phi^-(D) = \phi^+(D),\ \phi^-(E) = \phi^+(E), & \\ (\beta^- \nabla \phi^- - \beta^+ \nabla \phi^+) \cdot \mathbf{n}(\overline{DE}) = 0, & \end{cases} \tag{2.5}$$

where $\mathbf{n}(\overline{DE})$ is the unit vector perpendicular to the line $\overline{DE}$.

Remark 2.1. The last three equations can be modified accordingly to generate the IFE space that can be used to handle an interface problem with (nonzero) constant interface jump conditions.

Now, we let $\phi_i(X)$ be the piecewise linear function described by (2.5) such that

$$\phi_i(x_j, y_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

for $1 \leq i, j \leq 3$. Then, we let $S_h(T)$ be the linear space of all the functions defined by (2.5), and call it the immersed finite element (IFE) space on an interface triangle T.

As usual, we need only define the nodal IFE basis functions in the reference triangle $\hat{T}$ with vertices $\hat{A}$, $\hat{B}$, and $\hat{C}$:

$$\hat{A} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \hat{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \hat{C} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The interface triangle T is related to the reference triangle by the usual affine mapping:

$$F(\hat{X}) = B + M\hat{X}, \qquad X = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \hat{X} = \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}.$$

Under this mapping, D becomes $\hat{D} = (0, \hat{y}_1)^T$, E becomes $\hat{E} = (1 - \hat{y}_2, \hat{y}_2)^T$ with $0 < \hat{y}_1, \hat{y}_2 < 1$, and

$$\phi(X) = \hat{\phi}(F^{-1}(X)) = \hat{\phi}(\hat{X}). \tag{2.6}$$

In the reference triangle $\hat{T}$, the basis function $\hat{\phi}$ has the following format:

$$\hat{\phi}(\hat{X}) = \begin{cases} \hat{\phi}^+(\hat{X}) = \hat{\phi}^+(\hat{A}) + a_1\hat{x} + a_2(\hat{y} - 1), & \hat{X} \in \hat{T}^+, \\ \hat{\phi}^-(\hat{X}) = \hat{\phi}(\hat{B}) + (\hat{\phi}(\hat{C}) - \hat{\phi}(\hat{B}))\hat{x} + b_2\hat{y}, & \hat{X} \in \hat{T}^-. \end{cases} \tag{2.7}$$

The continuity of $\hat{\phi}$ at $\hat{D}$ and $\hat{E}$ leads to

$$a_2(\hat{y}_1 - 1) - b_2\hat{y}_1 = \hat{\phi}(\hat{B}) - \hat{\phi}(\hat{A}), \tag{2.8}$$

$$a_1(1-\hat{y}_2)+a_2(\hat{y}_2-1)-b_2\hat{y}_2=\hat{\phi}(\hat{B})-\hat{\phi}(\hat{A})+(\hat{\phi}(\hat{C})-\hat{\phi}(\hat{B}))(1-\hat{y}_2). \quad (2.9)$$

The flux jump condition $(\beta^-\nabla\phi^- - \beta^+\nabla\phi^+)\cdot\mathbf{n}(\overline{DE}) = 0$ becomes

$$(\beta^-\nabla\hat{\phi}^- - \beta^+\nabla\hat{\phi}^+)\cdot\hat{\mathbf{n}}(\overline{DE})=0, \text{ with } \hat{\mathbf{n}}(\overline{DE})=M^{-1}(\mathbf{n}(\overline{DE})-B)=\begin{pmatrix}\hat{n}_1\\ \hat{n}_2\end{pmatrix},$$

which leads to

$$a_1\hat{n}_1+a_2\hat{n}_2-\rho\hat{n}_2 b_2=\rho(\hat{\phi}_i(\hat{C})-\hat{\phi}_i(\hat{B}))\hat{n}_1, \quad (2.10)$$

with

$$\rho=\beta^-/\beta^+.$$

Note that vector $(\alpha, -1)^T$ is parallel to the normal vector of the line segment $\overline{\hat{D}\hat{E}}$ with

$$\alpha=\frac{\hat{y}_2-\hat{y}_1}{1-\hat{y}_2}, \qquad \hat{y}_2\neq 1.$$

We then let $(\hat{\alpha}, -1)$ be the vector parallel to $\mathbf{n}(\overline{DE})$ and assume that

(H_3): $\alpha\hat{\alpha}\geq 0$ and $\hat{\alpha}=O(\alpha)$ when $|\alpha|\to\infty$.

It is important to notice that assumption (H_3) above does cover many typical situations including perhaps the most significant application of the IFE space in which a Cartesian partition $\mathcal{T}_h$ is used. In this case, the affine transformation can be chosen such that $\alpha=\hat{\alpha}$ and the assumption (H_3) can then be satisfied naturally.

Theorem 2.1. *The function $\phi(x, y)$ defined by (2.5) in an interface triangle T is uniquely decided by its values at the three vertices of T.*

Proof. We carry out the proof only for $\hat{\phi}$. Equations (2.8), (2.9), and (2.10) form a linear system about a_1, a_2, and b_2 whose matrix is

$$\mathcal{A}=\begin{pmatrix}0 & \hat{y}_1-1 & -\hat{y}_1\\ 1-\hat{y}_2 & \hat{y}_2-1 & -\hat{y}_2\\ \hat{\alpha} & -1 & \rho\end{pmatrix}, \quad (2.11)$$

and the right hand side of this system is

$$\mathbf{r}=\begin{pmatrix}\hat{\phi}_i(\hat{B})-\hat{\phi}_i(\hat{A})\\ \hat{\phi}_i(\hat{B})-\hat{\phi}_i(\hat{A})+(\hat{\phi}_i(\hat{C})-\hat{\phi}_i(\hat{B}))(1-\hat{y}_2)\\ \rho(\hat{\phi}_i(\hat{C})-\hat{\phi}_i(\hat{B}))\hat{\alpha}\end{pmatrix}.$$

Then

$$\det(\mathcal{A})=(1-\hat{y}_2)(\hat{y}_1+\hat{\alpha}\alpha+\rho(1-\hat{y}_2)(1+\alpha))$$

$$= (1 - \hat{y}_2)\left(\hat{y}_1 + \hat{\alpha}\alpha + \rho\left(1 - \frac{\hat{y}_1 + \alpha}{1 + \alpha}\right)(1 + \alpha)\right)$$

$$= (1 - \hat{y}_2)(\hat{y}_1 + \hat{\alpha}\alpha + \rho(1 - \hat{y}_1))$$

$$\geq (1 - \hat{y}_2)(\hat{\alpha}\alpha + \min\{1, \rho\}) > 0$$

because $0 \leq \hat{y}_1 \leq 1$ and $0 \leq \hat{y}_2 < 1$. This implies that this linear system has a unique solution, and the function $\hat{\phi}$ or ϕ is uniquely determined by the jump conditions and its values at vertices of the triangle. ■

We would like to make the following remarks about the IFE space $S_h(T)$.

- The proof of Theorem 2.1 describes a way to construct the nodal basis functions in an interface triangle, and $\phi_i(X)$, $i = 1, 2, 3$ form a basis for $S_h(T)$.
- From the proof we can see that $\phi^-(x, y) = \phi^+(x, y)$ when $\rho = 1$, i.e., when the coefficient does not have a jump, the functions in $S_h(T)$ become the usual linear polynomials. In this case, $S_h(T)$ reduces to the standard linear finite element space.
- When $\Gamma \cap T$ is a straight line, the function $\phi(x, y)$ defined by (2.5) is continuous in T and therefore is in $H^1(T)$.

We now turn to the discussion on the properties of the IFE functions.

Theorem 2.2. *For an interface triangle T, every function $\phi \in S_h(T)$ satisfies the flux jump condition on $\Gamma \cap T$ exactly in the following weak sense*:

$$\int_{\Gamma \cap T} (\beta^- \nabla \phi^- - \beta^+ \nabla \phi^+) \cdot \mathbf{n}_\Gamma ds = 0.$$

Proof. For any $\phi \in S_h(T)$, it is obvious that $\phi^s \in H^2(T^s)$, $s = -, +$. Also, because ϕ is a piecewise linear polynomial satisfying (2.5), Green's formula leads to

$$\int_{\Gamma \cap T} (\beta^- \nabla \phi^- - \beta^+ \nabla \phi^+) \cdot \mathbf{n}_\Gamma ds = -\int_{\overline{DE}} (\beta^- \nabla \phi^- - \beta^+ \nabla \phi^+) \cdot \mathbf{n}_{\overline{DE}} ds = 0.$$ ■

Theorem 2.3. *For the three functions $\phi_i(X)$, $i = 1, 2, 3$ defined above, we have*

$$\phi_1(X) + \phi_2(X) + \phi_3(X) = 1.$$

Proof. Again, we need only show that this is true for $\hat{\phi}_i$ whose parameters in (2.7) are a_{1i}, a_{2i}, b_{2i}, $i = 1, 2, 3$. Using the linear system determining the parameters a_1, a_2, b_2 in the proof of the Theorem 2.1, we have

$$a_{1i} = \begin{cases} \dfrac{\alpha(1-\rho)}{\hat{y}_1 + \hat{\alpha}\alpha + \rho(1-\hat{y}_1)}, & i = 1, \\ \dfrac{-(1+\hat{\alpha}\alpha)\rho + (1+\alpha)(\rho-1)\hat{y}_2}{\hat{y}_1 + \hat{\alpha}\alpha + \rho(1-\hat{y}_1)}, & i = 2, \\ \dfrac{\rho + \alpha(-1+\rho+\hat{\alpha}\rho) - (1+\alpha)(\rho-1)\hat{y}_2}{\hat{y}_1 + \hat{\alpha}\alpha + \rho(1-\hat{y}_1)}, & i = 3, \end{cases}$$

$$a_{2i} = \begin{cases} \dfrac{\hat{\alpha}\alpha + \rho}{\hat{y}_1 + \hat{\alpha}\alpha + \rho(1-\hat{y}_1)}, & i = 1, \\ \dfrac{-(1+\hat{\alpha}\alpha)\rho + \hat{\alpha}(1+\alpha)(\rho-1)\hat{y}_2}{\hat{y}_1 + \hat{\alpha}\alpha + \rho(1-\hat{y}_1)}, & i = 2, \\ -\dfrac{\hat{\alpha}(-1+\rho)(-\alpha+(1+\alpha)\hat{y}_2)}{\hat{y}_1 + \hat{\alpha}\alpha + \rho(1-\hat{y}_1)}, & i = 3, \end{cases}$$

$$b_{2i} = \begin{cases} \dfrac{\hat{\alpha}\alpha + 1}{\hat{y}_1 + \hat{\alpha}\alpha + \rho(1-\hat{y}_1)}, & i = 1, \\ \dfrac{-1 - \hat{\alpha}(-1+\rho+\alpha\rho) + \hat{\alpha}(1+\alpha)(\rho-1)\hat{y}_2}{\hat{y}_1 + \hat{\alpha}\alpha + \rho(1-\hat{y}_1)}, & i = 2, \\ -\dfrac{\hat{\alpha}(1+\alpha)(-1+\rho)(-1+\hat{y}_2)}{\hat{y}_1 + \hat{\alpha}\alpha + \rho(1-\hat{y}_1)}, & i = 3. \end{cases}$$

A simple calculation can show that

$$\begin{aligned} &a_{j1} + a_{j2} + a_{j3} = 0, \qquad j = 1, 2, \\ &b_{21} + b_{22} + b_{23} = 0, \end{aligned}$$

which together with (2.7) leads to the result of this lemma. ■

In the discussion below, we need another assumption on the partition $\mathcal{T}_h$.

(H_4): The family of partitions $\mathcal{T}_h$ with $h > 0$ is regular. (See Definition 3.4.1 of [20].)

Theorem 2.4. *There exists a constant C such that for any interface triangle $T \in \mathcal{T}_h$ and $X \in T$ we have*

$$|\phi_i(X)| \le C, \tag{2.12}$$

$$\|\nabla \phi_i(X)\| \le Ch^{-1}. \tag{2.13}$$

Proof. Obviously, (2.12) follows from the boundedness of the parameters a_1, a_2, b_2 in (2.7). From the proof of the previous lemma, we can see that these coefficients are linear combinations of the following functions:

$$\frac{1}{\hat{y}_1 + \hat{\alpha}\alpha + \rho(1-\hat{y}_1)}, \qquad \frac{\alpha}{\hat{y}_1 + \hat{\alpha}\alpha + \rho(1-\hat{y}_1)}, \qquad \frac{\hat{\alpha}\alpha}{\hat{y}_1 + \hat{\alpha}\alpha + \rho(1-\hat{y}_1)}.$$

Under our assumptions, it is easy to see that these functions of α and $\hat{\alpha}$ are bounded. This implies the boundedness of $\hat{\phi}_i$; hence the boundedness of ϕ_i, $i = 1, 2, 3$. As for the second inequality, note that

$$\nabla\hat{\phi}_i(\hat{X}) = \begin{cases} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, & \hat{X} \in \hat{T}^+ \\ \begin{pmatrix} \hat{\phi}_i(\hat{C}) - \hat{\phi}_i(\hat{B}) \\ b_2 \end{pmatrix}, & \hat{X} \in \hat{T}^-, \end{cases}$$

and $\nabla\phi_i = M^{-T}\nabla\hat{\phi}_i$. Because the partition is regular, we have $\|M^{-T}\| \le Ch^{-1}$. Because $\|\nabla\hat{\phi}_i\|$ is bounded, we finally have

$$\|\nabla\phi_i\| \le Ch^{-1}. \qquad \blacksquare$$

Now we use the partition $\mathcal{T}_h$ to define an immersed finite element (IFE) space $S_h(\Omega)$. We first define a piecewise linear nodal basis function $\phi(x, y)$ for each node $(x_N, y_N)^t$ of $\mathcal{T}_h$ such that $\phi(x_N, y_N) = 1$ but zero at other nodes, and $\phi|_T \in S_h(T)$ for any triangle $T \in \mathcal{T}_h$. Here $S_h(T)$ is the usual space of linear polynomials when T is a noninterface triangle, or the immersed finite element space on T introduced above when T is an interface triangle. Then we define $S_h(\Omega)$ as the span of these nodal basis functions, and it is easy to see that $S_h(\Omega)$ has the following properties:

- For a partition $\mathcal{T}_h$, the IFE space $S_h(\Omega)$ has the same number of nodal basis functions as that formed by the usual linear polynomials.
- For a partition $\mathcal{T}_h$ fine enough, most of its triangles are noninterface triangles, and most of the nodal basis functions of the IFE space $S_h(\Omega)$ are just the usual linear nodal basis functions except for few nodes in the vicinity of the interface Γ.
- For any $\phi \in S_h(\Omega)$, we have
$$\phi|_{\Omega\backslash\Omega'} \in H^1(\Omega\backslash\Omega'), \tag{2.14}$$
where Ω' is the union of interface triangles.

3. ERROR ESTIMATES FOR INTERPOLATION APPROXIMATIONS

For any $T \subset \Omega$, we let

$$PW_p^m(T) = \{u | u|_{T^s} \in W_p^m(T^s),\ s = -, +\}, \qquad p \ge 1, m = 0, 1, 2,$$

$$PH_{int}^2(T) = \left\{u \in C(T),\ u|_{T^s} \in H^2(T^s),\ s = -, +, \left[\beta\frac{\partial u}{\partial \mathbf{n}_\Gamma}\right] = 0 \quad \text{on } \Gamma \cap T\right\},$$

$$PC_{int}^m(T) = \left\{u \in C(T),\ u|_{T^s} \in C^m(T^s),\ s = -, +, \left[\beta\frac{\partial u}{\partial \mathbf{n}_\Gamma}\right] = 0 \quad \text{on } \Gamma\right\}.$$

As usual, we define $PH^m(T) = PW_2^m(T)$. Obviously, we have $PC_{int}^2(T) \subset PH_{int}^2(T)$. Also, for any function $u \in PW_p^m(T)$, we let

$$\|u\|_{m,p,T}^2 = \|u\|_{m,p,T^-}^2 + \|u\|_{m,p,T^+}^2, \tag{3.15}$$

where $\|\cdot\|_{m,p,T^s}$ is the norm of $W_p^m(T^s)$, $s = -, +$. Seminorms of $PW_p^m(T)$ can be defined accordingly by

$$|u|^2_{m,p,T} = |u|^2_{m,p,T^-} + |u|^2_{m,p,T^+}. \tag{3.16}$$

When $p = 2$, we will drop p from the notation of the norms, e.g., we will use

$$\|u\|_{m,T} = \|u\|_{m,2,T}.$$

Similar definitions can be introduced for $PH^m(\Omega)$, $PH^2_{int}(\Omega)$.

In this section, we assume that the interface curve Γ and the partition $\mathcal{T}_h$ chosen satisfy the following assumptions:

(H_5): The interface curve Γ is defined by a piecewise C^2 function, and the partition $\mathcal{T}_h$ is formed such that the subset of Γ in any interface triangle is C^2.

(H_6): The interface Γ is smooth enough so that $PC^3_{int}(T)$ is dense in $PH^2_{int}(T)$ for any interface triangle T of $\mathcal{T}_h$.

The results of [21, 22] on the transmission problems show that (H_6) will hold if Γ is sufficiently smooth.

For a function $u \in PH^2_{int}(T)$, $T \in \mathcal{T}_h$, we let $I_{h,T}u \in S_h(T)$ be its interpolant such that $I_{h,T}u(X) = u(X)$ when X is a vertex of T. For an interface triangle T with vertices A, B, C, we have

$$I_{h,T}u(X) = u(A)\phi_1(X) + u(B)\phi_2(X) + u(C)\phi_3(X).$$

Accordingly, for a function $u \in PH^2_{int}(\Omega)$, *we let* $I_hu \in S_h(\Omega)$ be its interpolation such that $I_hu|_T = I_{h,T}(u|_T)$ for any $T \in \mathcal{T}_h$. The purpose of this section is to derive error estimates for the interpolation of $u \in PH^2_{int}(\Omega)$.

Recall that the error estimate of I_hu in any noninterface triangle T is well known; see for example [20]:

$$\|I_hu - u\|_{0,T} + h\|I_hu - u\|_{1,T} \leq Ch^2\|u\|_{2,T}.$$

Therefore, we focus the following discussion on interface triangles. For an arbitrary interface triangle (see Fig. 3) we let T^* be the subset in T enclosed by the interface Γ and the line segment $\overline{DE}$, and let

$$T^{*s} = T^s \backslash T^*, \qquad s = -, +.$$

For any point $\tilde{A} \in \Gamma$, we let $\tilde{A}_\perp$ be the orthogonal projection of $\tilde{A}$ onto $\overline{DE}$ (see Fig. 3). We will use the following four matrices:

$$N^-(\tilde{A}) = \begin{pmatrix} n_y(\tilde{A})^2 + \rho n_x(\tilde{A})^2 & (\rho - 1)n_x(\tilde{A})n_y(\tilde{A}) \\ (\rho - 1)n_x(\tilde{A})n_y(\tilde{A}) & n_x(\tilde{A})^2 + \rho n_y(\tilde{A})^2 \end{pmatrix},$$

$$N^+(\tilde{A}) = \begin{pmatrix} n_y(\tilde{A})^2 + \tilde{\rho} n_x(\tilde{A})^2 & (\tilde{\rho} - 1)n_x(\tilde{A})n_y(\tilde{A}) \\ (\tilde{\rho} - 1)n_x(\tilde{A})n_y(\tilde{A}) & n_x(\tilde{A})^2 + \tilde{\rho} n_y(\tilde{A})^2 \end{pmatrix}, \qquad \tilde{\rho} = \frac{1}{\rho},$$

$$N^-_{\overline{DE}} = \begin{pmatrix} \bar{n}_y^2 + \rho\bar{n}_x^2 & (\rho - 1)\bar{n}_x\bar{n}_y \\ (\rho - 1)\bar{n}_x\bar{n}_y & \bar{n}_x^2 + \rho\bar{n}_y^2 \end{pmatrix},$$

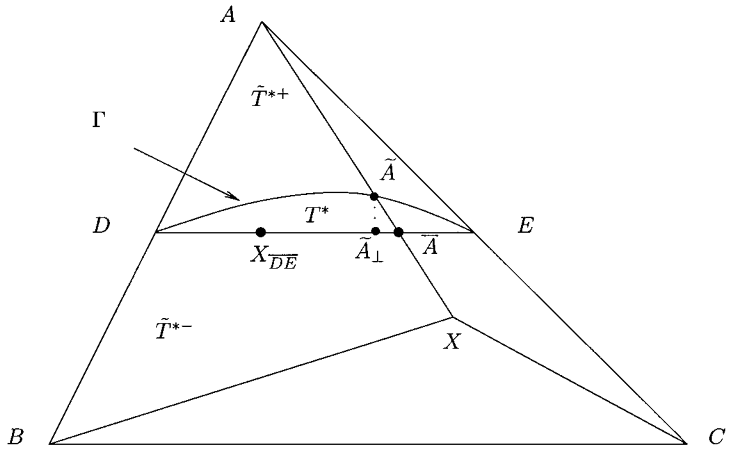

FIG. 3. An interface triangle with no obscure point. A point $X \in T^{*-}$ is connected to the three vertices by line segments.

$$N^{+}_{\overline{DE}} = \begin{pmatrix} \bar{n}_y^2 + \tilde{\rho}\bar{n}_x^2 & (\tilde{\rho} - 1)\bar{n}_x\bar{n}_y \\ (\tilde{\rho} - 1)\bar{n}_x\bar{n}_y & \bar{n}_x^2 + \tilde{\rho}\bar{n}_y^2 \end{pmatrix},$$

where $\mathbf{n}(\tilde{A}) = (n_x(\tilde{A}), n_y(\tilde{A}))^T$ is the unit normal vector of Γ at $\tilde{A}$, and $\mathbf{n}(\overline{DE}) = (\bar{n}_x, \bar{n}_y)^T$ is the unit normal vector of $\overline{DE}$. These matrices relate the left and right limit values of the gradient of a function at the point $\tilde{A}$ of interface Γ or a point on the line segment $\overline{DE}$. For example, we can easily verify that, for any function $u(x, y)$ satisfying the interface jump conditions (1.3) and (1.4), we have

$$\nabla u^+(\tilde{A}) = N^-(\tilde{A})\nabla u^-(\tilde{A}).$$

Because $\Gamma \cap T$ is a C^2 curve, when the partition $\mathcal{T}_h$ is fine enough, we can introduce a local coordinate system centered at point D with one axis in the direction of $\overline{DE}$. For any point $(x, y)^T$, let (ξ, η) be its coordinates in this local coordinate system. Then we have

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_D \\ y_D \end{pmatrix} + \begin{pmatrix} \cos(\theta_{DE}) & -\sin(\theta_{DE}) \\ \sin(\theta_{DE}) & \cos(\theta_{DE}) \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{3.17}$$

where $(x_D, y_D)^T$ is the coordinates of point D and θ_{DE} is the angle between $\overline{DE}$ and the x axis. As in [23], we can assume that Γ has the following equation in this local system:

$$\eta = \phi(\xi), \qquad \xi \in [0, |\overline{DE}|], \tag{3.18}$$

with

$$|\phi(\xi)| \leq Ch^2, \tag{3.19}$$

$$|\phi'(\xi)| \leq Ch. \tag{3.20}$$

From now on, if necessary, for any point P, we will use

$$\begin{pmatrix} x_P \\ y_P \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \xi_P \\ \eta_P \end{pmatrix}$$

to denote its coordinates in the $x - y$ and $\xi - \eta$ systems, respectively.

Lemma 3.1. *There exist constants $C > 0$ and $h_0 > 0$ such that for all $0 \leq h \leq h_0$ and any point $\tilde{A} \in \Gamma \cap T$, we have*

$$\|\tilde{A} - \tilde{A}_\perp\| \leq Ch^2. \tag{3.21}$$

$$\|N^s_{\overline{DE}} - N^s(\tilde{A})\| \leq Ch, \qquad s = -, +, \tag{3.22}$$

where $T \in \mathcal{T}_h$ is an arbitrary interface element.

Proof. We can prove these only in the local coordinate system because the transformation (3.17) preserves the vector length. In the local system, we have

$$\tilde{A} = \begin{pmatrix} \tilde{\xi} \\ \phi(\tilde{\xi}) \end{pmatrix}, \qquad \tilde{A}_\perp = \begin{pmatrix} \tilde{\xi} \\ 0 \end{pmatrix}.$$

Hence (3.21) is just the consequence of (3.19). Also, we have

$$\mathbf{n}(\overline{DE}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \mathbf{n}(\tilde{A}) = \frac{1}{\sqrt{1 + (\phi'(\tilde{\xi}))^2}} \begin{pmatrix} -\phi'(\tilde{\xi}) \\ 1 \end{pmatrix}.$$

Then, by (3.20), we have

$$\|\mathbf{n}(\overline{DE}) - \mathbf{n}(\tilde{A})\| \leq Ch,$$

which together with definition of $N^s_{\overline{DE}}$ and $N^s(\tilde{A})$, $s = -, +$, lead to (3.22). ∎

We call a point $X = (x, y)^T$ in an interface triangle $T \in \mathcal{T}_h$ an *obscure point* if one of the three line segments passing through X and the vertices of T intersects the interface more than once. Without loss of generality, we discuss an interface triangle that does not contain any obscure point because the arguments used below can be readily extended to handle interface triangles with obscure points.

For any function $u \in PH^2_{int}(T)$, the error estimates for $I_{h,T}u$ is obtained by estimates over the three subsets T^{*-}, T^{*+}, and T^* of T. The key issue is to establish suitable multipoint Taylor expansions for functions in $S_h(T)$ and $PC^3(T)$.

We start with the estimation on T^{*-}. Let $X = (x, y)^T$ be a point in T^{*-}. Without loss of generality, we can assume that line segments $\overline{XB}$ and $\overline{XC}$ do not intersect with the interface and $\overline{DE}$, whereas line segment $\overline{XA}$ meets Γ and $\overline{DE}$ at $\tilde{A}$ and $\bar{A}$, respectively (see Fig. 3). Also, we assume that

$$\tilde{A} = \tilde{t}A + (1 - \tilde{t})X = (\tilde{x}, \tilde{y})^T$$

$$\bar{A} = \bar{t}A + (1 - \bar{t})X = (\bar{x}, \bar{y})^T.$$

Lemma 3.2. *Given a real number r, a two-dimensional vector $\mathbf{q}$, a point $X \in T^{*-}$, and a point $X_{\overline{DE}} \in \overline{DE}$, there exits a function $v \in S_h(T)$ such that $v(X) = r$, $\nabla v(X) = \mathbf{q}$, and*

$$\begin{aligned} 0 = \mathbf{q} \cdot ((A - X)\phi_1(X) + (B - X)\phi_2(X) + (C - X)\phi_3(X)) \\ + (N_{\overline{DE}}^{-} - I)\mathbf{q} \cdot (A - X)(1 - \tilde{t})\phi_1(X) + (N_{\overline{DE}}^{-} - I)\mathbf{q} \cdot (\tilde{A} - X_{\overline{DE}})\phi_1(X). \end{aligned} \tag{3.23}$$

Proof. Let

$$v(Y) = \begin{cases} v^-(Y), & Y \in T^-, \\ v^+(Y), & Y \in T^+ \end{cases}$$

be a function in $S_h(T)$. Because $v(Y)$ is piecewise linear, $v(X) = r$, $\nabla v(X) = \mathbf{q}$ uniquely determine $v^-(Y)$. Then the interface conditions $v^-(D) = v^+(D)$, $v^-(E) = v^+(E)$, and $\beta\text{-}(\partial v^-/\partial \mathbf{n}_{\overline{DE}}) = \beta^+(\partial v^+/\partial \mathbf{n}_{\overline{DE}})$ uniquely determine $v^+(Y)$.

Because $v^-(Y)$ is a linear polynomial, we have

$$v(B) = v^-(B) = r + q \cdot (B - X),$$

$$v(C) = v^-(C) = r + q \cdot (C - X).$$

Similarly, because $v^+(Y)$ is a linear polynomial and the jump conditions satisfied by $v(Y)$ give $\nabla v^+(Y) = N_{\overline{DE}}^{-}\nabla v^-(Y)$, we have

$$\begin{aligned} v(A) = v^+(A) &= v^+(X_{\overline{DE}}) + \nabla v^+(X_{\overline{DE}}) \cdot (A - X_{\overline{DE}}) \\ &= v^-(X_{\overline{DE}}) + N_{\overline{DE}}^{-}\nabla v^-(X_{\overline{DE}}) \cdot (A - X_{\overline{DE}}) \\ &= v^-(X) + \mathbf{q} \cdot (X_{\overline{DE}} - X) + N_{\overline{DE}}^{-}\mathbf{q} \cdot (A - X_{\overline{DE}}) \\ &= r + \mathbf{q} \cdot (A - X) + (N_{\overline{DE}}^{-} - I)\mathbf{q} \cdot (A - \tilde{A}) + (N_{\overline{DE}}^{-} - I)\mathbf{q} \cdot (\tilde{A} - X_{\overline{DE}}) \\ &= r + \mathbf{q} \cdot (A - X) + (N_{\overline{DE}}^{-} - I)\mathbf{q} \cdot (A - X)(1 - t) + (N_{\overline{DE}}^{-} - I)\mathbf{q} \cdot (\tilde{A} - X_{\overline{DE}}). \end{aligned}$$

Then, from these expansions of $v(Y)$ at the vertices of T, we have

$$\begin{aligned} v(X) = I_{h,T}v(X) &= v(A)\phi_1(X) + v(B)\phi_2(X) + v(C)\phi_3(X) \\ &= r\sum_{i=1}^{3} \phi_i(X) + \mathbf{q} \cdot ((A - X)\phi_1(X) + (B - X)\phi_2(X) + (C - X)\phi_3(X)) \\ &\quad + (N_{\overline{DE}}^{-} - I)\mathbf{q} \cdot (A - X)(1 - \tilde{t})\phi_1(X) + (N_{\overline{DE}}^{-} - I)\mathbf{q} \cdot (\tilde{A} - X_{\overline{DE}})\phi_1(X), \end{aligned}$$

and the proof is finished because $v(X) = r$ and $\Sigma_{i=1}^{3} \phi_i(X) = 1$. ∎

Lemma 3.3. *For any $u \in PC_{int}^3(T)$, $X \in T^{*-}$, and $X_{\overline{DE}} \in \overline{DE}$ we have*

$$I_{h,T}u(X) - u(X) = (N^-(\tilde{A}) - N^-_{\overline{DE}})\nabla u(X)\cdot(A - X)\phi_1(X)(1-\tilde{t}) - (N^-_{\overline{DE}} - I)\nabla u(X)$$
$$\cdot(\tilde{A} - X_{\overline{DE}})\phi_1(X) + (1-\tilde{t})(N^-(\tilde{A}) - I)\int_0^1 \frac{d\nabla u^-}{dt}(t\tilde{A} + (1-t)X)\cdot(A - X)dt\ \phi_1(X)$$
$$+ \int_0^{\tilde{t}} (1-t)\frac{d^2u}{dt^2}(tA + (1-t)X)dt\ \phi_1(X) + \int_{\tilde{t}}^1 (1-t)\frac{d^2u}{dt^2}(tA + (1-t)X)dt\ \phi_1(X)$$
$$+ \int_0^1 (1-t)\frac{d^2u}{dt^2}(tB + (1-t)X)dt\ \phi_2(X) + \int_0^1 (1-t)\frac{d^2u}{dt^2}(tC + (1-t)X)dt\ \phi_3(X). \tag{3.24}$$

Proof. Since $t \mapsto u(tB + (1-t)X)$ is a C^2 function, we have

$$u(B) = u(X) + \int_0^1 \frac{du}{dt}(tB + (1-t)X)dt$$
$$= u(X) + \nabla u(X)\cdot(B - X) + \int_0^1 (1-t)\frac{d^2u}{dt^2}(tB + (1-t)X)dt. \tag{3.25}$$

Similarly, we have

$$u(C) = u(X) + \int_0^1 \frac{du}{dt}(tC + (1-t)X)dt$$
$$= u(X) + \nabla u(X)\cdot(C - X) + \int_0^1 (1-t)\frac{d^2u}{dt^2}(tC + (1-t)X)dt. \tag{3.26}$$

Using the jump condition across the interface, we have

$$\nabla u^+(\tilde{A}) = \begin{pmatrix} n_y(\tilde{A})^2 + \rho n_x(\tilde{A})^2 & (\rho - 1)n_x(\tilde{A})n_y(\tilde{A}) \\ (\rho - 1)n_x(\tilde{A})n_y(\tilde{A}) & n_x(\tilde{A})^2 + \rho n_y(\tilde{A})^2 \end{pmatrix}\nabla u^-(\tilde{A}) = N^-(\tilde{A})\nabla u^-(\tilde{A}).$$

Then, we have

$$u(A) = u(X) + \int_0^1 \frac{du}{dt}(tA + (1-t)X)dt$$
$$= u(X) + \int_0 \frac{du}{dt}(tA + (1-t)X)dt + \int_{\tilde{t}}^1 \frac{du}{dt}(tA + (1-t)X)dt$$

$$= u(X) - \nabla u^-(\tilde{A}) \cdot (A - X)(1 - \tilde{t}) + \nabla u(X) \cdot (A - X)$$

$$+ \int_0^{\tilde{t}} (1-t) \frac{d^2u}{dt^2}(tA + (1-t)X)dt + \nabla u^+(\tilde{A}) \cdot (A - X)(1 - \tilde{t})$$

$$+ \int_{\tilde{t}}^1 (1-t) \frac{d^2u}{dt^2}(tA + (1-t)X)dt$$

$$= u(X) + \nabla u(X) \cdot (A - X) + (N^-(\tilde{A}) - I)\nabla u^-(\tilde{A}) \cdot (A - X)(1 - \tilde{t})$$

$$+ \int_0^{t} (1-t) \frac{d^2u}{dt^2}(tA + (1-t)X)dt + \int_{\tilde{t}}^1 (1-t) \frac{d^2u}{dt^2}(tA + (1-t)X)dt = u(X)$$

$$+ \nabla u(X) \cdot (A - X) + (N^-(\tilde{A}) - I)\nabla u(X) \cdot (A - X)(1 - \tilde{t})$$

$$+ (1 - \tilde{t})(N^-(\tilde{A}) - I) \int_0^1 \frac{d\nabla u^-}{dt}(t\tilde{A} + (1-t)X) \cdot (A - x)dt$$

$$+ \int_0^{t} (1-t) \frac{d^2u}{dt^2}(tA + (1-t)X)dt + \int_{\tilde{t}}^1 (1-t) \frac{d^2u}{dt^2}(tA + (1-t)X)dt. \qquad (3.27)$$

Then

$$I_{h,T}u(X) = u(A)\phi_1(X) + u(B)\phi_2(X) + u(C)\phi_3(X)$$

$$= u(X) \sum_{i=1}^{3} \phi_i(X) + \nabla u(X) \cdot ((A - X)\phi_1(X) + (B - X)\phi_2(X)$$

$$+ (C - X)\phi_3(X)) + (N^-(\tilde{A}) - I)\nabla u^-(X) \cdot (A - X)(1 - \tilde{t})\phi_1(X)$$

$$+ (1 - \tilde{t})(N^-(\tilde{A}) - I) \int_0^1 \frac{d\nabla u^-}{dt}(t\tilde{A} + (1-t)X) \cdot (A - X)dt \ \phi_1(X)$$

$$+ \int_0^{t} (1-t) \frac{d^2u}{dt^2}(tA + (1-t)X)dt \ \phi_1(X)$$

$$+ \int_{\tilde{t}}^1 (1-t) \frac{d^2u}{dt^2}(tA + (1-t)X)dt \ \phi_1(X)$$

$$+ \int_0^1 (1-t) \frac{d^2u}{dt^2}(tB + (1-t)X)dt \ \phi_2(X)$$

$$+\int_0^1 (1-t)\frac{d^2u}{dt^2}(tC+(1-t)X)dt\,\phi_3(X). \tag{3.28}$$

Now let $v \in S_h(T)$ be such that

$$v(Y) = \begin{cases} v^+(Y), & Y \in T^+, \\ v^-(Y), & Y \in T^- \end{cases}$$

with first degree polynomials $v^-(Y)$, $v^+(Y)$ determined by

$$\begin{cases} v^-(X) = u^-(X), & \nabla v^-(X) = \nabla u^-(X), \\ v^+(D) = v^-(D), & v^+(E) = v^-(E), \qquad \beta^+ \dfrac{\partial v^+(\bar{A})}{\partial \mathbf{n}(\overline{DE})} = \beta^- \dfrac{\partial v^-(\bar{A})}{\partial \mathbf{n}(\overline{DE})}. \end{cases}$$

Then by Lemma 3.2, we have

$$u(X) = u(X)\sum_{i=1}^{3}\phi_i(X) + \nabla u(X)\cdot((A-X)\phi_1(X) + (B-X)\phi_2(X) + (C-X)\phi_3(X))$$
$$+ (N^-_{\overline{DE}} - I)\nabla u(X)\cdot(A-X)(1-\tilde{t})\phi_1(X) + (N^-_{\overline{DE}} - I)\nabla u(X)\cdot(\tilde{A} - X_{\overline{DE}})\phi_1(X). \tag{3.30}$$

Finally, (3.24) follows from (3.28) and (3.30). ∎

Theorem 3.1. *There exists a constant C such that*

$$\|I_{h,T}u - u\|_{0,T^{*-}} \le Ch^2\|u\|_{2,T} \tag{3.31}$$

for any $u \in PH^2_{int}(T)$, where T is an arbitrary interface triangle.

Proof. Because of (H_6), we need only show that (3.31) is true for any $u \in PC^3_{int}(T)$.

We proceed by estimating the L^2 norms for each term Q_i, $i = 1, 2, \ldots, 7$ on the righthand side of (3.24). By Lemma 3.1 and Theorem 2.4, we have the following estimate for the L^2 norms of the first two terms by letting $X_{\overline{DE}} = \tilde{A}_\perp$:

$$\|Q_1\|_{0,T^{*-}} + \|Q_2\|_{0,T^{*-}} = \|(N^-(\tilde{A}) - N^-_{\overline{DE}})\nabla u(X)\cdot(A-X)\phi_1(X)(1-\tilde{t})\|_{0,T^{*-}}$$
$$+ \|(N^-_{\overline{DE}} - I)\nabla u^-(X)\cdot(\tilde{A} - \tilde{A}_\perp)\phi_1(X)\|_{0,T^{*-}} \le Ch^2\|u\|_{1,T^{*-}} \le Ch^2\|u\|_{2,T^{*-}}.$$

For the third term, we first note that

$$\frac{d\nabla u^-}{dt}(t\tilde{A} + (1-t)X)\cdot(A-X) = u_{xx}(\xi,\eta)(\tilde{x}-x)^2$$
$$+ 2u_{xy}(\xi,\eta)(\tilde{x}-x)(\tilde{y}-y) + u_{yy}(\xi,\eta)(\tilde{y}-y)^2,$$

with $\xi = t\tilde{x} + (1-t)x$, $\eta = t\tilde{y} + (1-t)y$. Then,

$$Q_3^2 \leq C(1-\tilde{t})^2\left(\int_0^1 [u_{xx}(\xi,\eta)(\tilde{x}-x)^2 + 2u_{xy}(\xi,\eta)(\tilde{x}-x)(\tilde{y}-y) + u_{yy}(\xi,\eta)(\tilde{y}-y)^2]dt\right)^2$$

$$\leq Ch^4(1-\tilde{t})^2\left(\int_0^1 [u_{xx}(\xi,\eta) + 2u_{xy}(\xi,\eta) + u_{yy}(\xi,\eta)]dt\right)^2$$

$$\leq Ch^4(1-\tilde{t})^2 \int_0^1 [u_{xx}^2(\xi,\eta) + u_{xy}^2(\xi,\eta) + u_{yy}^2(\xi,\eta)]\, dt,$$

where C stands for a generic constant whose value changes from line to line. Therefore,

$$\begin{aligned}\|Q_3\|_{0,T^{*-}}^2 &= \int_{T^{*-}} Q_3^2 dX \\ &\leq Ch^4(1-\tilde{t})^2 \int_{T^{*-}} \int_0^1 [u_{xx}^2(\xi,\eta) + u_{xy}^2(\xi,\eta) + u_{yy}^2(\xi,\eta)]\, dtdX \\ &\leq Ch^4 \int_{T^-} [u_{xx}^2(\xi,\eta) + u_{xy}^2(\xi,\eta) + u_{yy}^2(\xi,\eta)]\, dX \\ &\leq Ch^4\|u\|_{2,T^-}^2,\end{aligned}$$

or

$$\|Q_3\|_{0,T^{*-}} \leq Ch^2\|u\|_{2,T^-}.$$

For the fourth term, we have

$$\begin{aligned}Q_4^2 \leq C&\left(\int_0^{\tilde{t}} (1-t)[u_{xx}(\xi,\eta)(x_A-x)^2 + 2u_{xy}(\xi,\eta)(x_A-x)(y_A-y)\right. \\ &\left. + u_{yy}(\xi,\eta)(y_A-y)^2]\, dt\right)^2 \\ \leq Ch^4&\left(\int_0^{\tilde{t}} (1-t)[u_{xx}(\xi,\eta) + 2u_{xy}(\xi,\eta) + u_{yy}(\xi,\eta)]dt\right)^2 \\ \leq Ch^4& \int_0^{\tilde{t}} (1-t)^2[u_{xx}^2(\xi,\eta) + u_{xy}^2(\xi,\eta) + u_{yy}^2(\xi,\eta)]\, dt,\end{aligned}$$

with $\xi = tx_A + (1-t)x$, $\eta = ty_A + (1-t)y$, $A = (x_A, y_A)^T$. Therefore,

$$\|Q_4\|_{T^{*-}}^2 \leq Ch^4 \int_{T^{*-}} \int_0^{\tilde{t}} (1-t)^2 [u_{xx}^2(\xi, \eta) + u_{xy}^2(\xi, \eta) + u_{yy}^2(\xi, \eta)] \, dt dX$$

$$\leq Ch^4 \int_{T^-} [u_{xx}^2(\xi, \eta) + u_{xy}^2(\xi, \eta) + u_{yy}^2(\xi, \eta)] \, dX \leq Ch^4 |u\|_{2,T^-}$$

or

$$\|Q_4\|_{T^{*-}} \leq Ch^2 \|u\|_{2,T^-}.$$

Similarly, we can show that

$$\|Q_5\|_{T^{*-}} \leq Ch^2 \|u\|_{2,T^+},$$
$$\|Q_6\|_{T^{*-}} \leq Ch^2 \|u\|_{2,T^-},$$
$$\|Q_7\|_{T^{*-}} \leq Ch^2 \|u\|_{2,T^-}.$$

Finally, (3.31) follows from the estimates for Q_i, $i = 1, 2, \ldots, 7$ above. ■

We now turn to the estimate of the H^1 norm on T^{*-}. In the following two lemmas, we let I_1, I_2, and I_3 be the integral terms in the expansions (3.27), (3.25), and (3.26), respectively.

Lemma 3.4. *For any $u \in PC_{int}^3(T)$, $X \in T^{*-}$, and $X_{\overline{DE}} \in \overline{DE}$, we have*

$$\frac{\partial (I_{h,T}u(X) - u(X))}{\partial s} = (N^-(\tilde{A}) - N_{\overline{DE}}^-)\nabla u(X)(A - \tilde{A}) \frac{\partial \phi_1(X)}{\partial s}$$
$$- (N_{\overline{DE}}^- - I)\nabla u(X)(\tilde{A} - X_{\overline{DE}}) \frac{\partial \phi_1(X)}{\partial s} + \left(I_1 \frac{\partial \phi_1}{\partial s} + I_2 \frac{\partial \phi_2}{\partial s} + I_3 \frac{\partial \phi_3}{\partial s} \right), \qquad s = x, y. \tag{3.32}$$

Proof. We give a proof only for the case in which $s = x$. The case in which $s = y$ can be carried out similarly. From (3.24) in Lemma 3.3, we have

$$\frac{\partial (I_{h,T}u(X) - u(X))}{\partial x} = \frac{\partial}{\partial x} [(N^-(\tilde{A}) - N_{\overline{DE}}^-)\nabla u(X)(A - \tilde{A})]\phi_1(X)$$
$$+ (N^-(\tilde{A}) - N_{\overline{DE}}^-)\nabla u(X)(A - \tilde{A}) \frac{\partial \phi_1(X)}{\partial x} - \frac{\partial}{\partial x} [(N_{\overline{DE}}^- - I)\nabla u(X)(\tilde{A} - X_{\overline{DE}})]\phi_1(X)$$
$$- (N_{\overline{DE}}^- - I)\nabla u(X)(\tilde{A} - X_{\overline{DE}}) \frac{\partial \phi_1(X)}{\partial x} + \left(\frac{\partial I_1}{\partial x} \phi_1(X) + \frac{\partial I_2}{\partial x} \phi_2(X) + \frac{\partial I_3}{\partial x} \phi_3(X) \right)$$
$$+ \left(I_1 \frac{\partial \phi_1}{\partial x} + I_2 \frac{\partial \phi_2}{\partial x} + I_3 \frac{\partial \phi_3}{\partial x} \right). \tag{3.33}$$

From the expansions (3.27), (3.25), and (3.26), we have

$$0 = \frac{\partial^2 u(X)}{\partial x^2}(x_A - x) + \frac{\partial^2 u(X)}{\partial x \partial y}(y_A - y) + \frac{\partial}{\partial x}[(N^-(\tilde{A}) - I)\nabla u(X)(A - \tilde{A})] + \frac{\partial I_1}{\partial x},$$

$$0 = \frac{\partial^2 u(X)}{\partial x^2}(x_B - x) + \frac{\partial^2 u(X)}{\partial x \partial y}(y_B - y) + \frac{\partial I_2}{\partial x},$$

$$0 = \frac{\partial^2 u(X)}{\partial x^2}(x_C - x) + \frac{\partial^2 u(X)}{\partial x \partial y}(y_C - y) + \frac{\partial I_3}{\partial x}.$$

Then

$$\frac{\partial I_1}{\partial x}\phi_1(X) + \frac{\partial I_2}{\partial x}\phi_2(X) + \frac{\partial I_3}{\partial x}\phi_3(X) = -[\mathbf{p}\cdot(A - X)\phi_1(X) + \mathbf{p}\cdot(B - X)\phi_2(X) + \mathbf{p}\cdot(C - X)\phi_3(X)] - \frac{\partial}{\partial x}[(N^-(\tilde{A}) - I)\nabla u(X)(A - \tilde{A})]\phi_1(X),$$

where

$$\mathbf{p} = \begin{pmatrix} \dfrac{\partial^2 u(X)}{\partial x^2} \\ \dfrac{\partial^2 u(X)}{\partial x \partial y} \end{pmatrix}.$$

Let $v(Y) \in S_h(T)$ be such that $\nabla v(X) = \mathbf{p}$. Then by Lemma 3.2, we have

$$0 = \mathbf{p}\cdot(A - X)\phi_1(x) + \mathbf{p}\cdot(B - X)\phi_2(x) + \mathbf{p}\cdot(C - X)\phi_3(x) + (N^-_{\overline{DE}} - I)\mathbf{p}\cdot(A - \tilde{A})\phi_1(X) + (N^-_{\overline{DE}} - I)\mathbf{p}\cdot(\tilde{A} - X_{\overline{DE}})\phi_1(X).$$

Hence

$$\frac{\partial I_1}{\partial x}\phi_1(X) + \frac{\partial I_2}{\partial x}\phi_2(X) + \frac{\partial I_3}{\partial x}\phi_3(X) = (N^-_{\overline{DE}} - I)\mathbf{p}\cdot(A - \tilde{A})\phi_1(X) + (N^-_{\overline{DE}} - I)\mathbf{p}\cdot(\tilde{A} - X_{\overline{DE}})\phi_1(X) - \frac{\partial}{\partial x}[(N^-(\tilde{A}) - I)\nabla u(X)(\tilde{A} - X)]\phi_1(X),$$

and

$$\frac{\partial(I_{h,T}u(X) - u(X))}{\partial x} = \frac{\partial}{\partial x}[(N^-(\tilde{A}) - N^-_{\overline{DE}})\nabla u(X)(A - \tilde{A})]\phi_1(X) + (N^-(\tilde{A}) - N^-_{\overline{DE}})\nabla u(X)(A - \tilde{A})\frac{\partial \phi_1(X)}{\partial x} - \frac{\partial}{\partial x}[(N^-_{\overline{DE}} - I)\nabla u(X)(\tilde{A} - X_{\overline{DE}})]\phi_1(X) - (N^-_{\overline{DE}} - I)\nabla u(X)(\tilde{A} - X_{\overline{DE}})\frac{\partial \phi_1(X)}{\partial x} + (N^-_{\overline{DE}} - I)\mathbf{p}$$

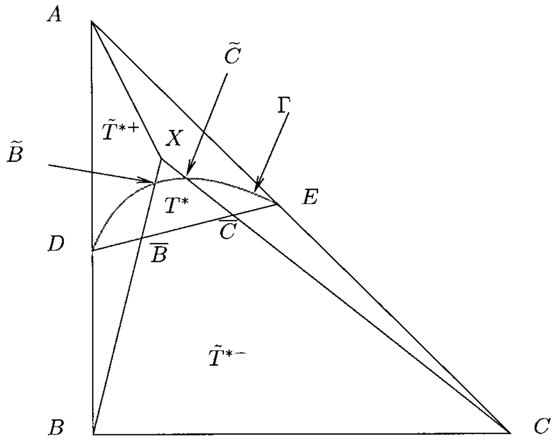

FIG. 4. A point $X \in T^{*+}$ is connected to the three vertices by line segments.

$$\cdot (A - \tilde{A})\phi_1(X) + (N^-_{\overline{DE}} - I)\mathbf{p} \cdot (\tilde{A} - X_{\overline{DE}})\phi_1(X) - \frac{\partial}{\partial x}[(N^-(\tilde{A}) - I)\nabla u(X)(A - \tilde{A})]\phi_1(X)$$

$$+ \left(I_1 \frac{\partial \phi_1}{\partial x} + I_2 \frac{\partial \phi_2}{\partial x} + I_3 \frac{\partial \phi_3}{\partial x}\right) = (N^-(\tilde{A}) - N^-_{\overline{DE}})\nabla u(X)(A - \tilde{A}) \frac{\partial \phi_1(X)}{\partial x}$$

$$- (N^-_{\overline{DE}} - I)\nabla u(X)(\tilde{A} - X_{\overline{DE}}) \frac{\partial \phi_1(X)}{\partial x} + (N^-_{\overline{DE}} - I)\mathbf{p} \cdot (A - \tilde{A})\phi_1(X) + (N^-_{\overline{DE}} - I)\mathbf{p}$$

$$\cdot (\tilde{A} - X_{\overline{DE}})\phi_1(X) + \left(I_1 \frac{\partial \phi_1}{\partial x} + I_2 \frac{\partial \phi_2}{\partial x} + I_3 \frac{\partial \phi_3}{\partial x}\right) + \frac{\partial}{\partial x}[(I - N^-_{\overline{DE}})\nabla u(X)$$

$$\times \ (A - X_{\overline{DE}})]\phi_1(X) = (N^-(\tilde{A}) - N^-_{\overline{DE}})\nabla u(X)(A - \tilde{A}) \frac{\partial \phi_1(X)}{\partial x}$$

$$- (N^-_{\overline{DE}} - I)\nabla U(X)(\tilde{A} - X_{\overline{DE}}) \frac{\partial \phi_1(X)}{\partial x} + \left(I_1 \frac{\partial \phi_1}{\partial x} + I_2 \frac{\partial \phi_2}{\partial x} + I_3 \frac{\partial \phi_3}{\partial x}\right). \quad \blacksquare$$

Theorem 3.2. *There exits a constant C such that*

$$\left\| \frac{\partial (I_{h,T}u - u)}{\partial s} \right\|_{0,T^{*-}} \le Ch\|u\|_{2,T}, \qquad s = x, y, \tag{3.34}$$

for any $u \in PH^2_{int}(T)$, where T is an arbitrary interface triangle.

Proof. Because of (H_6), we need only show that (3.34) is true for any $u \in PC^3_{int}(T)$. The result follows by letting $X_{\overline{DE}} = \tilde{A}_\perp$ in (3.32) and applying arguments similar to those used in the proof of Theorem 3.1. Note that (2.13) in Theorem 2.4 has to be used here. ■

The estimation on T^{*+} is rather similar. We state the results in the following four lemmas. Please see Fig. 4 for the notations involved. In particular, we let

$$\tilde{B} = \tilde{t}_B B + (1 - \tilde{t}_B)X = (\tilde{x}_B, \tilde{y}_B)^T,$$

$$\bar{B} = \bar{t}_B B + (1 - \bar{t}_B)X = (\bar{x}_B, \bar{y}_B)^T,$$

$$\tilde{C} = \tilde{t}_C C + (1 - \tilde{t}_C)X = (\tilde{x}_C, \tilde{y}_C)^T,$$

$$\bar{C} = \bar{t}_C C + (1 - \bar{t}_C)X = (\bar{x}_C, \bar{y}_C)^T.$$

Lemma 3.5. *Given a real number r, a two-dimensional vector $\mathbf{q}$, a point $X \in T^{*+}$, and two points $X_{\overline{DE},B} \in \overline{DE}$, $X_{\overline{DE},C} \in \overline{DE}$, there exits a function $v \in S_h(T)$ such that $v(X) = r$, $\nabla v(X) = \mathbf{q}$, and*

$$\begin{aligned} 0 = \mathbf{q} \cdot ((A - X)\phi_1(x) + (B - X)\phi_2(x) + (C - X)\phi_3(x)) + (N^+_{\overline{DE}} - I)\mathbf{q} \cdot (\tilde{B} - X_{\overline{DE},B})\phi_2(X) \\ + (N^+_{\overline{DE}} - I)\mathbf{q}(B - X)(1 - \tilde{t}_B)\phi_2(X) + (N^+_{\overline{DE}} - I)\mathbf{q} \cdot (\tilde{C} - X_{\overline{DE},C})\phi_3(X) \\ + (N^+_{\overline{DE}} - I)\mathbf{q}(C - X)(1 - \tilde{t}_C)\phi_3(X). \end{aligned} \tag{3.35}$$

Lemma 3.6. *For any $u \in PC^2(T)$, $X \in T^{*+}$, we have*

$$\begin{aligned} I_{h,T}u(X) - U(X) = (N^+(\tilde{B}) - N^+_{\overline{DE}})\nabla u^+(X) \cdot (B - X)(1 - \tilde{t}_B)\phi_2(X) + (N^+(\tilde{C}) \\ - N^+_{\overline{DE}})\nabla u^+(X) \cdot (C - X)(1 - \tilde{t}_C)\phi_3(X) - (N^+_{\overline{DE}} - I)\nabla u(X) \cdot (\tilde{B} - B_\perp)\phi_2(X) \\ - (N^+_{\overline{DE}} - I)\nabla u(X) \cdot (\tilde{C} - C_\perp)\phi_3(X) + \int_0^1 (1 - t)\, \frac{d^2u}{dt^2}\,(tA + (1 - t)X)dt\ \phi_1(X) \\ + \int_0^{\tilde{t}_B} (1 - t)\, \frac{d^2u}{dt^2}\,(tB + (1 - t)X)dt\ \phi_2(X) + \int_{\tilde{t}_B}^1 (1 - t)\, \frac{d^2u}{dt^2}\,(tB + (1 - t)X)dt\ \phi_2(X) \\ + \int_0^{\tilde{t}_C} (1 - t)\, \frac{d^2u}{dt^2}\,(tC + (1 - t)X)dt\ \phi_3(X) + \int_{\tilde{t}_C}^1 (1 - t)\, \frac{d^2u}{dt^2}\,(tC + (1 - t)X)dt\ \phi_3(X) \\ + (N^+(\tilde{B}) - I)\left(\int_0^1 \frac{d\nabla u^+(t\tilde{B} + (1 - t)X)}{dt}\,dt\right) \cdot (B - X)(1 - \tilde{t}_B)\phi_2(X) + (N^+(\tilde{C}) - I) \\ \times \left(\int_0^1 \frac{d\nabla u^+(t\tilde{C} + (1 - t)X)}{dt}\,dt\right) \cdot (C - X)(1 - \tilde{t}_C)\phi_3(X). \end{aligned} \tag{3.36}$$

Theorem 3.3. *There exits a constant C such that*

$$\|I_{h,T}u - u\|_{0,T^{*+}} \leq Ch^2\|u\|_{2,T}, \tag{3.37}$$

for any $u \in PH^2_{int}(T)$, where T is an arbitrary interface triangle.

Theorem 3.4. *There exits a constant C such that*

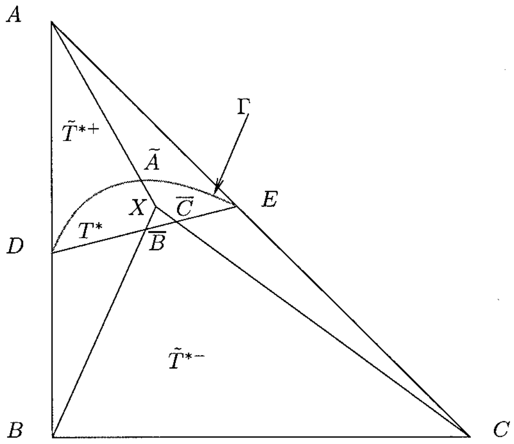

FIG. 5. A point $X \in T^*$ is connected to the three vertices by line segments.

$$\left\| \frac{\partial I_{h,T}u - u}{\partial s} \right\|_{0,T^{*+}} \leq Ch\|u\|_{2,T}, \qquad s = x, y,$$

for any $u \in PH^2_{int}(T)$, where T is an arbitrary interface triangle.

Similar multipoint expansions can be established on T^*. Please see Fig. 5 for the notations involved. In particular, we let

$$\tilde{A} = \tilde{t}_A A + (1 - \tilde{t}_A)X = (\tilde{x}_A, \tilde{y}_A),$$
$$\bar{B} = \bar{t}_B B + (1 - \bar{t}_B)X = (\bar{x}_B, \bar{y}_B)^T,$$
$$\bar{C} = \bar{t}_C C + (1 - \bar{t}_C)X = (\bar{x}_C, \bar{y}_C)^T.$$

Lemma 3.7. *Given a real number r, a two-dimensional vector $\mathbf{q}$, a point $X \in T^*$, and a point $X_{\overline{DE}} \in \overline{DE}$, there exits a function $v \in S_h(T)$ such that $v(X) = r$, $\nabla v(X) = \mathbf{q}$, and*

$$\begin{aligned} 0 = \mathbf{q} \cdot ((A - X)\phi_1(X) + (B - X)\phi_2(X)(+(C - X)\phi_3(X)) + (N^+_{\overline{DE}} - I)\mathbf{q}(\tilde{A} - X_{\overline{DE}})\phi_1(X) \\ +(I - N^+_{\overline{DE}})\mathbf{q} \cdot (A - X)(1 - \tilde{t}_A)\phi_1(X) + (N^+_{\overline{DE}} - I)\mathbf{q} \cdot (B - X)(1 - \bar{t}_B)\phi_2(X) \\ +(N^+_{\overline{DE}} - I)\mathbf{q} \cdot (C - X)(1 - \bar{t}_C)\phi_3(X). \end{aligned} \tag{3.38}$$

Lemma 3.8. *For any $u \in PC^3_{int}(T)$ and $X \in T^*$ we have*

$$\begin{aligned} I_{h,T}u(X) - u(X) = (N^-(\tilde{A}) - I)\nabla u(X) \cdot (A - X)(1 - \tilde{t}_A)\phi_1(X) - (I - N^+_{\overline{DE}})\nabla u(X) \\ \cdot(A - X)(1 - \tilde{t}_A)\phi_1(X) - (N^+_{\overline{DE}} - I)\nabla u(X) \cdot (B - X)(1 - \bar{t}_B)\phi_2(X) \\ - (N^+_{\overline{DE}} - I)\nabla u(X) \cdot (C - X)(1 - \bar{t}_C)\phi_3(X) - (N^+_{\overline{DE}} - I)\nabla u(X)(\tilde{A} - \tilde{A}_\perp)\phi_1(X) \end{aligned}$$

$$+(1-\tilde{t}_A)(N^-(\tilde{A})-I)\int_0^1 \frac{d\nabla u^-}{dt}(t\tilde{A}+(1-t)X)\cdot(A-X)dt\ \phi_1(X)$$

$$+\int_0^{\tilde{t}_A}(1-t)\frac{d^2u}{dt^2}(tA+(1-t)X)dt\ \phi_1(X)+\int_{\tilde{t}_A}^1(1-t)\frac{d^2u}{dt^2}(tA+(1-t)X)dt\ \phi_1(X)$$

$$+\int_0^1(1-t)\frac{d^2u}{dt^2}(tB+(1-t)X)dt\ \phi_2(X)+\int_0^1(1-t)\frac{d^2u}{dt^2}(tC+(1-t)X)dt\ \phi_3(X). \quad (3.39)$$

We now come to the error estimates on T^*, which are quite different from those obtained on T^{*s}, $s=-,+$.

Theorem 3.5. *For any $p>1$, there exists a constant C such that*

$$\|I_{h,T}u-u\|_{0,T^*}\le Ch^2\|u\|_{2,T}+Ch^{(5/2)-(3/p)}\|u\|_{1,p,T^*}, \quad (3.40)$$

for any $u\in PH^2_{int}(T)$, where T is an arbitrary interface triangle.

Proof. The proof is similar to that for Theorem 3.1. We need only derive estimates for the first five terms in (3.39). For the first term, we have

$$\begin{aligned}\|Q_1\|_{0,T^*}&\le Ch\left(\int_{T^*}\|\nabla u(X)\|^2dX\right)^{1/2}\\&\le Ch\left(\left[\int_{T^*}dX\right]^{1/q}\left[\int_{T^*}\|\nabla u(X)\|^{2p'}\,dX\right]^{1/p'}\right)^{1/2}\left(\frac{1}{q}+\frac{1}{p'}=1,\ 1<p',\ q<\infty\right)\\&=Ch\ mes(T^*)^{1/2q}\left(\int_{T^*}\|\nabla u(X)\|^{2p'}\,dX\right)^{1/(2p')}\\&\le Ch^{(5/2)-(3/p)}\left(\int_{T^*}\|\nabla u(X)\|^{p}\,dX\right)^{1/p}\quad(p=2p',\ mes(T^*)\le Ch^3)\\&=Ch^{(5/2)-(3/p)}\|u\|_{1,p,T^*}.\end{aligned}$$

Again, C above is a generic constant, and we have used the relationship $1/q=1-(2/p)$. Similarly, we can show that

$$\|Q_i\|_{0,T^*}\le Ch^{(5/2)-(3/p)}\|u\|_{1,p,T^*},\qquad i=2,3,\ldots,5.$$

Putting these together we have (3.40). ■

Similarly, we have

Theorem 3.6. *There exits a constant C such that*

$$\left\| \frac{\partial (I_{h,T} u - u)}{\partial s} \right\|_{0,T^*} \le Ch \|u\|_{2,T} + Ch^{(3/2)-(3/p)} \|u\|_{1,p,T^*}, \qquad s = x, y,$$

for any $u \in PH^2_{int}(T)$, where T is an arbitrary interface triangle.

We now derive the error estimates for the interpolation $I_h u$ in $S_h(\Omega)$. Let $\mathcal{T}_{h,int}$ denote the set of all the interface triangles, and let Ω' be the subset of Ω formed by the union of all the interface triangles.

Theorem 3.7. *There exists a constant C such that*

$$\|I_h u - u\|_{0,\Omega} \le Ch^2 \|u\|_{2,\Omega}, \tag{3.41}$$

$$\left\| \frac{\partial (I_h u - u)}{\partial s} \right\|_{0,\Omega} \le Ch \|u\|_{2,\Omega}, \quad s = x, y, \tag{3.42}$$

for any $u \in PH^2_{int}(\Omega)$ and $h > 0$ small enough.

Proof. From Theorems 3.1, 3.3, and 3.5, we have

$$\begin{aligned}
\|I_h u - u\|^2_{0,\Omega'} &= \sum_{T \in \mathcal{T}_{h,int}} \|I_{h,T} u - u\|^2_{0,T} \\
&\le Ch^4 \sum_{T \in \mathcal{T}_{h,int}} \|u\|^2_{2,T} + C \sum_{T \in \mathcal{T}_{h,int}} h^{5-(6/p)} \|u\|^2_{1,p,T^*} \\
&= Ch^4 \|u\|^2_{2,\Omega'} + Ch^{5-(6/p)} \sum_{T \in \mathcal{T}_{h,int}} \|u\|^2_{1,p,T^*} \\
&= Ch^4 \left(\|u\|^2_{2,\Omega'} + \sum_{T \in \mathcal{T}_{h,int}} \|u\|^2_{1,6,T^*} \right) \qquad \text{(letting } p = 6\text{)} \\
&\le Ch^4 (\|u\|^2_{2,\Omega'} + \|u\|^2_{1,6,\Omega'}) \\
&\le Ch^4 (\|u\|^2_{2,\Omega'} + \|u\|^2_{2,\Omega'}) \\
&= Ch^4 \|u\|^2_{2,\Omega'},
\end{aligned}$$

where, following the same argument used in [5], we also have used the following estimate from [24]:

$$\|u\|^2_{1,p,\Omega'} \le Cp \|u\|^2_{2,\Omega'},$$

in which C is a constant independent of $p \in [2, \infty)$. The estimate (3.41) of this theorem then follows by combining the above estimate and the estimate from the standard finite element interpolation theory:

$$\|I_h u - u\|^2_{0,\Omega/\Omega'} \leq Ch^4\|u\|^2_2.$$

Similar derivations can be carried out to obtain (3.42). ■

4. NUMERICAL EXAMPLES

We now present some numerical results to illustrate features of the IFE space in this article. Errors in both the IFE interpolant and the IFE solution to an interface problem will be given. The error estimation for this finite element method will be provided in a forthcoming article. See [8] for the numerical examples generated from the finite volume element method based on this IFE space. See [6] for numerical results generated by the finite element method based on a rectangular IFE space. In [2, 3], we also reported some numerical results for certain nonlinear interface problems in axial-symmetric three-dimensional domains.

Because of its simplicity, we only present results obtained by using IFE method based on Cartesian partitions in the rectangular domain $\Omega = (-1, 1) \times (-1, 1)$. The interface curve Γ is a circle with radius $r_0 = \pi/6.28$, which separates Ω into two subdomains Ω^- and Ω^+ with

$$\Omega^- = \{(x, y) : x^2 + y^2 \leq r_0^2\}.$$

First, we show numerical results for the IFE interpolant $I_h u$ of a test function

$$u(x, y) = \begin{cases} \dfrac{r^\alpha}{\beta^-}, & \text{if } r \leq r_0, \\ \dfrac{r^\alpha}{\beta^-} + \left(\dfrac{1}{\beta^-} - \dfrac{1}{\beta^+}\right) r_0^\alpha, & \text{otherwise}, \end{cases} \tag{4.43}$$

with $\alpha = 3$ and

$$\beta(x, y) = \begin{cases} \beta^-, & (x, y) \in \Omega^-, \\ \beta^+, & (x, y) \in \Omega^+. \end{cases}$$

Here

$$r = \sqrt{x^2 + y^2},$$

with the domain Ω and the curve Γ sketched in Fig. 6 together with a typical partition for our numerical results.

Table I contains actual errors of the IFE interpolant $I_h u$ with various partition sizes h for the coefficient function:

$$\beta(x, y) = \begin{cases} 1, & (x, y) \in \Omega^-, \\ 2, & (x, y) \in \Omega^+. \end{cases}$$

By simple calculations, we can easily see that the data in this table satisfy

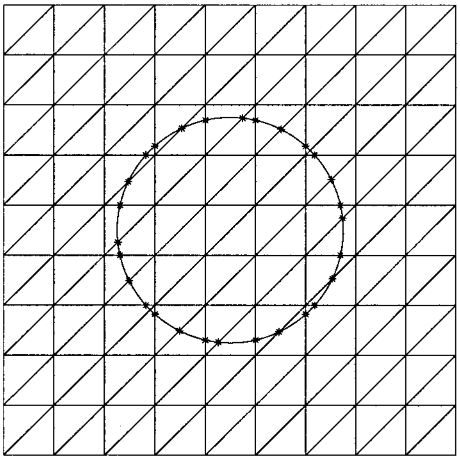

FIG. 6. A typical grid for the boundary value problem.

$$\|I_h u - u\|_0 \approx \frac{1}{4} \|I_{\hat{h}} u - u\|_0,$$

$$|I_h u - u|_1 \approx \frac{1}{2} |I_{\hat{h}} u - u|_1,$$

for $h = \hat{h}/2$. Using linear regression, we can also see that the data in this table obey

$$\|I_h u - u\|_0 \approx 0.855\ h^{1.997},$$
$$|I_h u - u|_1 \approx 1.070\ h^{1.004},$$

which clearly indicates that the interpolant converges to u with convergence rates $O(h^2)$ and $O(h)$ in the L^2 norm and H^1 norm, respectively, as predicted by Theorem 3.7.

Table II contains actual errors of the IFE interpolant $I_h u$ with various partition size h for the coefficient function with a larger jump:

$$\beta(x, y) = \begin{cases} 1, & (x, y) \in \Omega^-, \\ 1000, & (x, y) \in \Omega^+. \end{cases}$$

TABLE I. Errors in the interpolant $I_h u$ when $\beta^- = 1$, $\beta^+ = 2$.

h	$\|I_h u - u\|_0$	$\|I_h u - u\|_1$
1/8	0.01340296502645	0.13305717054982
1/16	0.00337158277650	0.06605013040974
1/32	0.00084503270234	0.03293429457757
1/64	0.00021161595036	0.01642432606383
1/128	0.00005293970281	0.00820432185076
1/256	0.00001323965088	0.00409991658967

TABLE II. Errors in the interpolant $I_h u$ when $\beta^- = 1$, $\beta^+ = 1000$.

h	$\|I_h u - u\|_0$	$\|I_h u - u\|_1$
1/8	0.00419883108135	0.04265805867861
1/16	0.00114533244388	0.02243935983649
1/32	0.00029604487946	0.01148491618700
1/64	0.00007569425803	0.00580755271391
1/128	0.00001909398803	0.00292084704814
1/256	0.00000479645980	0.00146463745501

Using linear regression again, we can see that

$$\|I_h u - u\|_0 \approx 0.256\ h^{1.959},$$

$$|I_h u - u|_1 \approx 0.332\ h^{0.975},$$

which are also in agreement with the error estimates given in Theorem 3.7.

Because the IFE space has an $O(h^2)$ (in L^2-norm) and an $O(h)$ (in H^1-norm) approximation capability, we naturally expect the finite element method based on this IFE space to perform accordingly. To confirm this numerically, we consider the interface value problem defined by (1.1)–(1.4) in which the boundary condition function $g(x, y)$ and the source term $f(x, y)$ are chosen such that for $\alpha = 3$ the function u given above is the exact solution in the domain Ω with the interface curve Γ defined before. All the IFE solutions presented here are generated with Cartesian partitions illustrated in Fig. 6. We refer readers to [1] for a comparison of the IFE method and the standard FE method using a comparable body fit partition.

Table III contains actual errors of the IFE solutions with various partition size h for the boundary value problem with the coefficient function:

$$\beta(x, y) = \begin{cases} 1, & (x, y) \in \Omega^-, \\ 2, & (x, y) \in \Omega^+. \end{cases}$$

We can easily see that the data in the second and third columns of this table satisfy

$$\|u_h - u\|_0 \approx \frac{1}{4}\,\|u_{\hat{h}} - u\|_0,$$

$$|u_h - u|_1 \approx \frac{1}{2}\,|u_{\hat{h}} - u|_1,$$

TABLE III. Numerical results for the case when $\beta^- = 1$, $\beta^+ = 2$.

h	$\|u_h - u\|_0$	$\|u_h - u\|_1$	$\|u_h - u\|_\infty$
1/8	0.01259368785855	0.13344867525901	0.00309617807508
1/16	0.00318515404195	0.06633277348461	0.00125071300736
1/32	0.00079688638455	0.03333299478264	0.00070955158740
1/64	0.00019975072596	0.01675883476638	0.00031533665223
1/128	0.00004995090290	0.00848995823388	0.00014497566355
1/256	0.00001247445372	0.00438231449652	0.00007395804432

TABLE IV. Numerical results for the case when $\beta^- = 1$, $\beta^+ = 1000$.

h	$\|u_h - u\|_0$	$\|u_h - u\|_1$
1/8	0.00501120556232	0.05453969221442
1/16	0.00135523677651	0.02707248382673
1/32	0.00035702301314	0.01342153161376
1/64	0.00007999830913	0.00643491056234
1/128	0.00001867669405	0.00322926125945
1/256	0.00000434992267	0.00166074411817

for $h = \hat{h}/2$. Using linear regression, we can also see that the data in this table obey

$$\|u_h - u\|_0 \approx 0.804\ h^{1.997},$$

$$|u_h - u|_1 \approx 1.026\ h^{0.987},$$

which indicates that the IFE solution u_h converges to the exact solution with convergence rates $O(h^2)$ and $O(h)$ in the L^2 norm and H^1 norm, respectively, and are in agreement with those error estimates for the IFE interpolant obtained in the previous section.

However, these numerical experiments indicate that the IFE method does not always have the second-order convergence in the L^∞ norm because the data in the fourth column of Table III obey

$$|u_h - u|_\infty \approx 0.0271\ h^{1.0696},$$

which clearly shows that the rate at which u_h converges to u is not $O(h^2)$. The question of the conditions under which the IFE solution can have a second order convergence in the L^∞ norm is still open.

The IFE method also works well for the case in which the coefficient function has a large jump, see Table IV. The errors in this group of computations obey

$$\|u_h - u\|_0 \approx 0.384\ h^{2.044},$$

$$|u_h - u|_1 \approx 0.446\ h^{1.013},$$

which again are in agreement with those error estimates for the IFE interpolant.

5. CONCLUSIONS

In this article, we have discussed an immersed finite element (IFE) space that can be used to solve interface problems of second-order elliptic partial differential equations. The partition of this IFE space can be formed without consideration of the interface location. If applicable, even a Cartesian partition can be used in this IFE space to solve a problem with a rather arbitrary interface. The IFE space is closely related to the standard finite element space formed by piecewise first-degree polynomials except for functions over interface triangles. Over an interface triangle, IFE functions are formed according to the jump conditions of the interface problem to be solved. We have employed the multipoint Taylor expansion technique to analyze the interpolation errors in the IFE space for functions in the Sobolev space related to the interface problems. It has been shown that the IFE space has an approximation capability similar to that of the standard linear finite element space. The estimates for the interpolation error obtained here are critical for deriving error estimates for the finite element (volume) solution to an interface problem based on this IFE space.

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