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# Topology optimization with adaptive mesh refinement

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## Abstract

We outline a robust method for topology optimization with adaptive mesh refinement and derefinement (AMR). Since the total volume fraction in topology optimization is usually modest, after a few initial iterations the domain of computation is largely void. It is inefficient to have many small elements in such regions, as these contribute significantly to the overall computational cost but little to the accuracy of computation and design. At the same time, we want high spatial resolution for accurate three-dimensional designs to avoid significant postprocessing or interpretation. AMR offers the possibility to balance these two requirements, but it has received little attention in the context of topology optimization. We will discus approaches by Costas and Alves [2] and Stainko [3]. Unfortunately, both approaches may lead to suboptimal designs that are mesh dependent. We extend these approaches to obtain a method that yields optimal designs, and we show experimentally that our improvements lead to designs that are equivalent to designs computed on uniform meshes at the finest level of refinement. Furthermore, we demonstrate significant reductions of run time by using AMR and efficient methods for the solution of the resulting large, linear systems, following Wang et al. [4].

## 1. Introduction

Topology optimization is a powerful structural optimization method that combines a numerical solution method, usually the finite element method, with an optimization algorithm to find the optimal material distribution inside a given domain. In designing the topology of a structure we determine which points in the domain should be material and which points should be void; see Bendsøe and Sigmund [1].

In topology optimization, problems are solved most commonly on fixed uniform meshes with a relatively large number of elements in order to achieve accurate designs. However, as void and solid regions appear in the design, it is more efficient to represent the holes with fewer large elements and the solid regions, especially the material surface, with more fine elements. Since the shape and position of holes and solid regions are initially unknown, the most economical mesh representation for the design is unknown a priori. Therefore, adaptive mesh refinement (AMR) is very suitable for topology optimization. *The purpose of AMR for topology optimization is to get the design that would be obtained on a uniformly fine mesh, but at a much lower computational cost by reducing the total number of elements and having fine elements only where and when necessary*.

Highly accurate designs on uniform meshes may require so many elements that the solution of the optimization problem becomes intractable. However, AMR leads to high resolution in the mesh only when and where necessary. This makes it possible to do highly accurate designs with a modest number of elements against a reasonable cost. Obviously, we do not want the use of AMR or the AMR procedure to alter the computed designs. However, there is a risk of this, since the mesh influences the computed deformations and sensitivities. So, the solutions from the finite element analysis using AMR must be as accurate as those obtained on a uniform fine mesh. Moreover, it must be the accurate solution and corresponding sensitivities obtained on the finest mesh that govern the design. If coarse mesh solutions drive or limit the design, suboptimal designs may result when designs optimal on a coarser mesh differ substantially from the optimal design on a (much) finer mesh; see Wang et al. [5]. The early work in this area, though leading to acceptable designs in specific instances, does not satisfy these properties. We propose simple but essential changes to these methodologies that lead to AMR based designs that are equivalent (up to some small tolerance) to designs on uniform fine meshes.

## 2. Topology Optimization in a nutshell

In topology optimization we solve for the material distribution in a given design domain $\Omega$. Here, we minimize the compliance of a structure under given loads as a function of the material distribution. To solve this problem numerically, we discretize the computational domain using finite elements, where we use a lower order interpolation for the density field (material distribution) than for the displacement field. We take the most common approach using trilinear interpolation for the displacement field and constant density in each element. The compliance minimization problem after finite element discretization is defined as

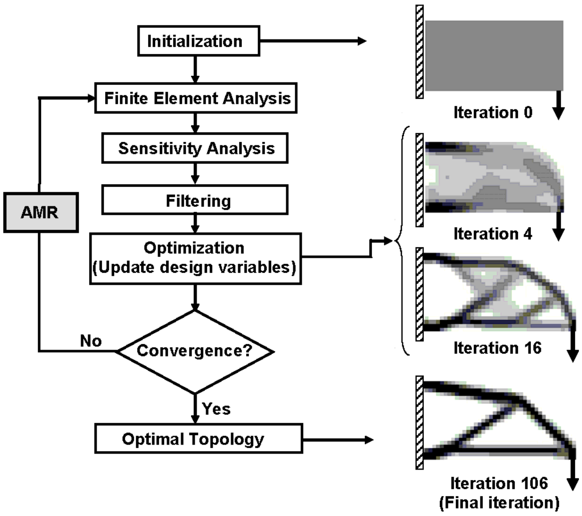


**Figure 1. Overview Topology Optimization Algorithm**

$$\min_{\forall e\ \rho_e \in [\rho_0, 1]} \mathbf{f}^T \mathbf{u}$$

$$\text{s.t.} \begin{cases} \mathbf{K}(\boldsymbol{\rho})\mathbf{u} = \mathbf{f} & \text{for } x \in \Omega \backslash \Omega_0, \\ \mathbf{u} = \mathbf{u}_0 & \text{for } x \in \Omega_0, \\ \sum_e \rho_e V_e \le V_0, & \end{cases}$$

with $\rho_e$ the density in element $e$, $\boldsymbol{\rho}$ the vector of element densities, $\mathbf{K}$ the stiffness matrix, which is a function of the element densities, $V_e$ the volume of element $e$, $V_0$ the maximum total volume (fraction) for the design, and $\Omega_0$ the part of the domain where the displacement is prescribed. We enforce a small positive lower bound, $\rho_0 = 10^{-3}$, on the element density to avoid singularity of the stiffness matrix.

The Solid Isotropic Material with Penalization method (SIMP) is used to make intermediate densities unfavorable; we define the elasticity tensor as a function of the element density, $E_e = \rho_e^p E_0$, where $p$ is the penalization parameter. With $p > 1$, intermediate densities provide little stiffness per (unit) volume. The common choice, $p = 3$, results in intermediate material properties satisfying the Hashin-Shtrikman bound for composite materials; see Bendsøe and Sigmund [1]. We apply continuation on $p$ to avoid problems with local minima, starting with $p = 1$ and slowly increasing $p$ as the design converges.

The general scheme for topology optimization using AMR is illustrated in Figure 1. Various optimization algorithms can be used for topology optimization. For this paper, we use Optimality Criteria (OC); see Bendsøe and Sigmund [1]. Dynamic mesh adaptation may be carried out before the finite element analysis.

## 3. Dynamic, Adaptive Mesh Refinement

Little research has been done in applying AMR to topology optimization. So, we briefly discuss two recent, important, papers in this area. The AMR method by Costa and Alves [2] goes through a predetermined, fixed sequence of optimizations and subsequent mesh refinements (they do not use derefinement), using (assuming) a converged solution on a `coarse mesh' to guide the refinement of that mesh and start the optimization on the next `fine mesh'. Coarse meshes and the solutions on these coarse meshes are never revisited or updated after generating the next finer mesh. The method aims at refining the coarse mesh design; after a fixed number of optimization steps on a given mesh, they refine all material elements and elements on the boundary between material and void. Stainko [3] follows a slightly different approach with respect to the refinements. Mesh refinement is done only along the material boundary as indicated by the (regularization) filter. So, elements completely inside a material region or a void region are not refined. These approaches share two important choices that may lead to problems. First, both approaches solve the design problem on a fixed mesh until convergence and then refine. After refinement on a given level, the mesh on that level remains fixed for the remainder of the optimization. Therefore, all further refinements are constrained by the converged coarser level solutions. This works well in terms of refining the design, but for many design problems the optimal solution on a uniform fine(st) mesh is quite different from the converged solution on a coarser mesh. In that case, mesh refinement based only on the coarser level solution will erroneously confine the solution on the finer mesh to a smooth version of the coarser level solution. Therefore, the approaches proposed in Costa and Alves [2] and Stainko [3] may lead to suboptimal designs; see Wang et al. [5]. Second, both approaches lack derefinement. This may lead to inefficiencies, having too many elements.

Next, we briefly describe the main ideas for our AMR strategy for topology optimization. Space restrictions prevent us from giving a detailed description, and we refer to Wang et al. [5] for details and implementation. We base our algorithmic choices on a set of requirements on AMR codes for topology optimization. As stated above, the purpose of AMR for topology optimization is to get the design that would be obtained on a uniform fine mesh, but at a much lower computational cost by reducing the total number of elements and having fine elements only where (and when) necessary.

First, since the finite element analysis and the computation of sensitivities drive the changes in material distribution, they should be as accurate as on the uniform fine mesh. Therefore, we need a fine mesh that covers at least the material region and the boundary. Since the void regions have negligible stiffness they do not influence the (intermediate) linear finite element solutions and sensitivity computations. So, we do not need a fine mesh inside the void regions. Hence, we use a refinement criterion similar to that of Costa and Alves \cite{Costa2003}. At this point we focus on refinement and derefinement for shape only. Therefore, we are conservative with respect to accuracy, and we expect that, in future implementations, good error indicators will lead to further efficiency gains, in particular because of derefinement in solid material regions. Second, the accurate computations on the finest level should drive the changes in the material distribution. This requires continual mesh adaptation so that computational results after refinements can drive updates to the material distribution, and designs are not confined by earlier coarse grid results. This also means that as the material region moves close to the boundary between fine and coarse(r) mesh, additional refinements allow for further evolution. Third, we need to ensure that the design can change sufficiently in between mesh updates. Therefore, we maintain a layer of additional refinements around the material region (in the void region) and carry out continual mesh adaptation. Due to the additional layer of refinements and continual mesh updates, the design can change arbitrarily following the fine grid computations and resulting sensitivities, and it is not confined by earlier coarse grid results. To ensure that the design accurately reflects the fine mesh computations, we allow rapid refinements of the mesh early on when voids and material regions (and hence the boundary) develop. Fourth, since the design can change substantially from its estimate on a coarse mesh, we may have fine elements in void regions. Those elements must be removed for efficiency, so we need derefinement. A hierarchical representation of adaptive meshes facilitates our strategy of continual mesh refinement and derefinement,.

## 4. A Numerical Experiment: Computing an Optimal Cantilever Beam

We compute the optimal design for the three-dimensional cantilever beam shown in Figure 2. Exploiting symmetry, we discretize only a quarter of the domain. We solve this problem first on a (fixed) uniform mesh with 128x32x32 B8 elements and then following our AMR strategy. The initial mesh for the AMR-based design has 64x16x16 B8 elements. The final results are shown in Figure 3, with the AMR solution on the left and the uniform mesh solution on the right. We measure the relative difference between two designs as follows

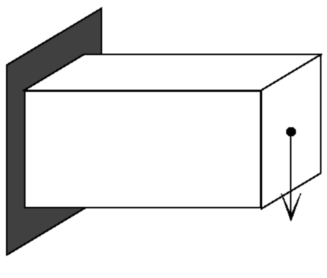

**Figure 2. 3D Cantilever beam with domain scale 2:1:1**

$$D\left(\rho^{(1)},\rho^{(2)}\right)=\frac{\int_{\Omega}\left|\rho^{(1)}-\rho^{(2)}\right|d\Omega}{\int_{\Omega}\rho^{(1)}d\Omega}. \tag{1}$$

The relative difference between these two designs, as defined by (1), is only 0.0909%.

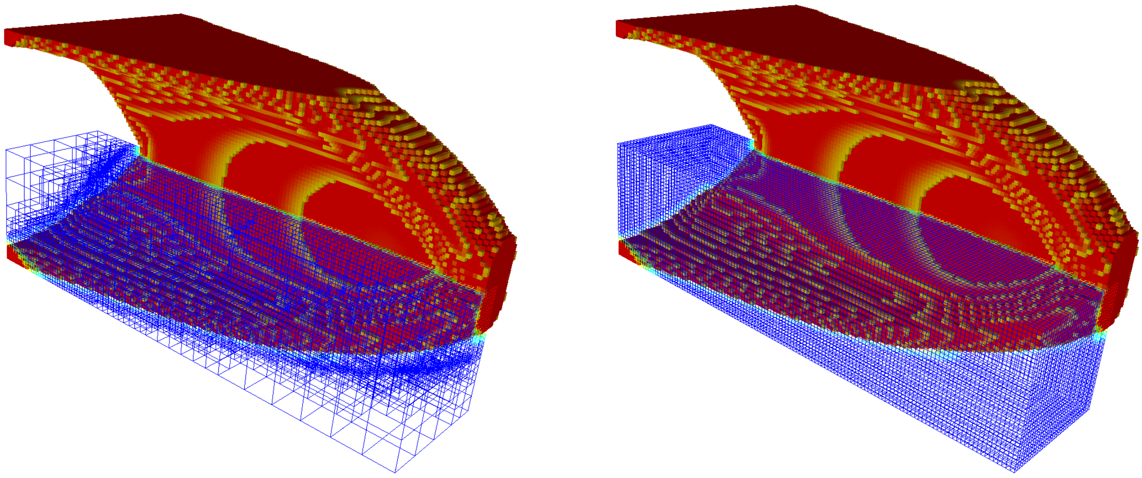

**Figure 3. Final AMR solution of the optimal 3D Cantilever beam problem and final solution on a fixed uniform mesh (128 x 32 x 32). The finest elements in both meshes are the same size. The relative difference in density distribution of both designs is only about 0.1%.**

We use the incomplete Cholesky-preconditioned, recycling minimum residual solver (RMINRES) proposed by Wang et al. [4] to solve the linear systems arising from the finite element discretization for a given material distribution. The dimensions of the linear systems of equations for the adaptive mesh are less than half of those for the uniform, fine mesh. The difference is even larger early in the optimization iteration. Moreover, the number of RMINRES iterations for the linear systems derived from the AMR mesh are smaller than those from the uniform, fine mesh, as the adaptive meshes lead to better conditioned linear systems. *In total, using AMR reduces the solution time roughly by a factor three.*

## 5. Conclusions

We have used AMR to reduce the total runtime for a three-dimensional topology optimization problem by a factor three while obtaining essentially the same design as on a fixed uniform fine mesh. Fast iterative solvers also play an important role in our approach. AMR provides a promising future research direction in topology optimization; especially important are efficiently updating preconditioners and combining refinement and derefinement for shape with refinement and derefinement based on a posteriori error estimates.

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