The Singular Value Decomposition

Let $A \in \mathbb{R}^{m\times n}$ be a nonzero matrix with rank r. Then there are orthogonal matrices $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ and diagonal matrix $\Sigma \in \mathbb{R}^{m\times n}$, $m \times n$

$$\Sigma = \begin{pmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & & & \sigma_r & & \\ & & & & 0 & \\ & & & & & 0 \; \ddots \end{pmatrix}$$

such that $A = U\Sigma V^T$ and

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$

This is called the singular value decomposition of A (SVD)

Alternatively, we can write (often useful)

$$A = \sum_{i=1}^{r} \sigma_i u_i v_i^T$$

SVD reveals many important properties of A

$\text{Range}(A) = \text{span}\{u_1, u_2, \cdots, u_r\}$
$\text{Null}(A) = \text{span}\{v_{r+1}, v_{r+2}, \cdots, v_n\}$
$\text{Range}(A^T) = \text{span}\{v_1, v_2, \cdots, v_r\}$
$\text{Null}(A^T) = \text{span}\{u_{r+1}, u_{r+2}, \cdots, u_m\}$

$\text{Rank}(A) = \dim(\text{Range}(A))$ is number of nonzero singular values (r here).

Note that from SVD: $\text{Range}(A) \perp \text{Null}(A^T)$
and $\text{Range}(A^T) \perp \text{Null}(A)$

Condensed (reduced) SVD

Let $A \in \mathbb{R}^{m\times n}$ be of rank r. Then there exist $\hat{U} \in \mathbb{R}^{m\times r}$, $\hat{\Sigma} \in \mathbb{R}^{r\times r}$, $\hat{V} \in \mathbb{R}^{n\times r}$, such that $\hat{U}, \hat{V}$ are isometries and $\hat{\Sigma}$ is a diagonal matrix with real positive entries $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ and

$$A = \hat{U}\hat{\Sigma}\hat{V}^T$$

Note $\hat{U}^T\hat{U} = I_r$ and $\hat{V}^T\hat{V} = I_r$

Let $x = V\xi \quad (\xi = V^T x)$

$= \sum_{i=1}^{n} v_i \xi_i$

$$Ax = U\Sigma V^T V\xi = U\Sigma\xi$$
$$= \sum_{i=1}^{r} u_i \sigma_i \xi_i$$

also

$$Ax = \left(\sum_{i=1}^{r} \sigma_i u_i v_i^T\right)\left(\sum_{j=1}^{n} v_j \xi_j\right)$$
$$= \sum_{i=1}^{r} \sigma_i u_i \xi_i$$

Vice versa, let $y = \sum_{i=1}^{r} u_i \xi_i$

then $y = Ax$, where $x = \sum_{i=1}^{r} v_i \xi_i/\sigma_i + x_0$

with $x_0 \in Null(A)$

$x_0 = \sum_{i=r+1}^{n} v_i \xi_i$ (ξ_i arbitrary)

=

Remember $\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$

$Av_1 = u_1\sigma_1 \Rightarrow \|A\|_2 \geq \sigma_1$

$x = VV^T x = v_1\xi_1 + v_2\xi_2 + \cdots + v_n\xi_n$

$(\xi = V^T x)$

$Ax = Av_1\xi_1 + \cdots + Av_n\xi_n$

$r = Rank(A)$

$= u_1\sigma_1\xi_1 + u_2\sigma_2\xi_2 + \cdots + u_r\sigma_r\xi_r + 0 + \cdots + 0$

$\|Ax\|_2^2 = (\sigma_1\xi_1)^2 + (\sigma_2\xi_2)^2 + \cdots + (\sigma_r\xi_r)^2$

$\leq \sigma_1^2(\xi_1^2 + \xi_2^2 + \cdots + \xi_r^2) \leq \sigma_1^2 \|x\|_2^2$

$\frac{\|Ax\|_2^2}{\|x\|_2^2} \leq \sigma_1^2$ for arbitrary x

Hence $\|A\|_2 = \sigma_1$

(Note V orthogonal $\rightarrow \|V^T x\|_2 = \|\xi\|_2 = \|x\|_2$)

$$\|A\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2\right)^{1/2}$$

$$\left(A \in \mathbb{R}^{m\times n} \rightarrow \left(\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2\right)^{1/2}\right)$$

$Q \in \mathbb{R}^{m\times m} \quad QQ^T = Q^TQ = I_m$

$A \in \mathbb{R}^{m\times n} = [a_1 \; a_2 \; \cdots \; a_n]$

$$\|A\|_F^2 = \sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij}^2\right) = \|a_1\|_2^2 + \|a_2\|_2^2 + \cdots + \|a_n\|_2^2$$

$$\|QA\|_F^2 = \|Qa_1\|_2^2 + \|Qa_2\|_2^2 + \cdots + \|Qa_n\|_2^2$$

$$= \|a_1\|_2^2 + \cdots + \|a_n\|_2^2 = \|A\|_F^2$$

From definition $\|A\|_F = \|A^T\|_F$.

$A^T = [\tilde{a}_1 \; \tilde{a}_2 \cdots \tilde{a}_m]$

$P \in \mathbb{R}^{n\times n}$, $PP^T = P^TP = I$

$$\|AP\|_F^2 = \|P^TA^T\|_F^2 = \|P^T\tilde{a}_1\|_2^2 + \cdots + \|P^T\tilde{a}_m\|_2^2$$

$$= \|\tilde{a}_1\|_2^2 + \cdots + \|\tilde{a}_m\|_2^2 = \|A^T\|_F^2 = \|A\|_F^2$$

So, for $Q^{m\times m}$, $P^{n\times n}$ orthogonal we have

$\|QAP\|_F = \|A\|_F$

Take $Q = U^T$, $P = V \rightarrow QAP = U^TU\Sigma V^TV = \Sigma$

and $\|A\|_F = \|QAP\|_F = \left(\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_r^2\right)^{1/2}$

Norms that have the property that multiplication by an orthogonal matrix (unitary in the complex case) does not change the norm are called <u>unitarily</u> <u>invariant</u> <u>norms</u>

This is a very useful property since orthogonal / unitary transformations have great stability properties and hence are very popular in algorithms

Derivation for Frobenius norm shows that any unitarily invariant norm must depend on the singular values only.

$$\|A\| = \|\Sigma\|$$

As discussed before the matrix 2-norm (also spectral norm) is also unitarily invariant.
Follows immediately from invariance of vector 2-norm (Euclidean)

$$\kappa_2(A) = \|A\|_2 \, \|A^{-1}\|_2 \quad \text{for } A \in \mathbb{R}^{n \times n}$$
A nonsingular

$$\text{Rank}(A) = n \rightarrow A = U \Sigma V^T$$

$$\text{where } \Sigma = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_n) \qquad \begin{matrix} (\sigma_i > 0) \\ \sigma_i \neq 0 \end{matrix}$$

$$\rightarrow A^{-1} = V \Sigma^{-1} U^T \qquad (AA^{-1} = A^{-1}A = I)$$

Singular values of A^{-1} are $\sigma_1^{-1}, \sigma_2^{-1}, \dots$
and

$$\sigma_n^{-1} \geq \sigma_{n-1}^{-1} \geq \dots \geq \sigma_1^{-1}$$

$$\text{Hence } \|A^{-1}\|_2 = \max_i (\sigma_i^{-1}) = \sigma_n^{-1}$$

$$(\text{or } \sigma_{min}^{-1})$$

So, $\kappa_2(A) = \|A\|_2 \, \|A^{-1}\|_2 = \frac{\sigma_1}{\sigma_n}$

Equivalently, we can now define the condition number of a nonsquare (rectangular) matrix.

$A \in \mathbb{R}^{m\times n}$, $A = U\Sigma V^T \rightarrow \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k \geq 0$

$\kappa_2(A) = \sigma_1 / \sigma_k$

where $k = \min(m, n)$.

We allow some (or all) of the singular values to be zero. This ~~[illegible]~~ would give $\kappa_2(A) = \infty$ (also for $n\times n$ case)

Consistent with nonuniqueness of solutions.

$\kappa_2(A)$ measures sensitivity of solutions (under small perturbations) in worst case.

$\kappa_2(A)$ arises in following case

Solve $Ax = u_1 \rightarrow x = v_1 \frac{1}{\sigma_1}$

Solve $A\hat{x} = u_1 + \varepsilon u_n \rightarrow \hat{x} = v_1 \frac{1}{\sigma_1} + v_n \frac{\varepsilon}{\sigma_n}$

(ε size of perturbation right hand side)

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$$\frac{\|\hat{x} - x\|_2}{\|x\|_2} = \frac{\|v_n \frac{\varepsilon}{\sigma_n}\|_2}{\|v_1 \frac{1}{\sigma_1}\|_2} = \varepsilon \frac{\sigma_1}{\sigma_n}$$

$(m > n)$

The above also holds for $A^{m\times n}$ if right hand side is consistent (in Range(A)).

$A^{m\times n}$, $m > n$
indep. columns

$\|A^TA\|_2 = \sigma_{max}(V\Sigma^T U^T U \Sigma V^T) = \sigma_1^2$

$\|(A^TA)^{-1}\|_2 = \sigma_{max}(V(\Sigma^T\Sigma)^{-1}V^T) = \sigma_n^{-2}$

$$\kappa_2(A^T A) = \frac{\sigma_1^2}{\sigma_n^2} = \left(\kappa_2(A)\right)^2$$

Hence, solving normal equations very sensitive even if A has modest condition number.

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Numerical Rank

The SVD reveals the rank of a matrix. However, we cannot compute (in general) the exact SVD.

Note that any (small) perturbation can make a singular matrix nonsingular.

$$\underset{m\times n}{A} = U\Sigma V^T \quad \text{where } \sigma_{r+1}, \sigma_{r+2}, \dots, \sigma_n = 0 \quad (\sigma_1, \dots, \sigma_r > 0)$$

$$\text{Let } \Delta = \varepsilon u_{r+1} v_{r+1}^T + \varepsilon u_{r+2} v_{r+2}^T + \cdots + \varepsilon u_n v_n^T$$

$$\text{Then } A + \Delta = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_r u_r v_r^T + \varepsilon u_{r+1} v_{r+1}^T + \cdots + \varepsilon u_n v_n^T$$

So, $(A+\Delta)$ is nonsingular for any $\varepsilon > 0$.

$\|\Delta\|_2 = \varepsilon$ (which can be made arbitrarily small)

However, a small perturbation cannot change the singular values by much.

Use concept of "numerical rank". The num. rank of a matrix is the number of singular values above a certain (appropriate) threshold.

For example, if the matrix has been perturbed only by rounding during input, a small multiple times machine precision times norm of the matrix is reasonable.

For example, the book suggests $\varepsilon = 10u\|A\|$.
Now let $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3 \geqslant \dots \geqslant \sigma_k \gg \varepsilon \geqslant \sigma_{k+1} \geqslant \cdots$.
We say the numerical rank of A is k.

In general, we prefer some ~~gap~~ gap in singular values (here $\sigma_k \gg \sigma_{k+1}$) so that choice for k rather than k+1 (or k-1) is not (too) arbitrary.

The accuracy of measured input data can also be used as an indication.

The key issue is whether, given the accuracy of the data, we can distinguish between two matrices of different rank (or actually distinguish two slightly different matrices).

Consider example above with $\sigma_1 = 1$, all entries accurate to double precision, $\varepsilon = 10^{-12}$, $\sigma_{k+1} = 10^{-16}$, ~~m~~
A perturbation of $O(u)$ of the coefficients will not change $\sigma_1 \dots \sigma_k$ significantly. We can safely say the matrix has (at least) rank k. However, claiming it has rank k+1 is dubious as a perturbation $O(u)$ can change this.

Theorem 4.2.15

$A \in \mathbb{R}^{m\times n}$, $\operatorname{rank}(A) = r > 0$, $A = U\Sigma V^T$ where $\sigma_1 \geq \cdots \geq \sigma_r > 0$

Define $A_k = U\Sigma_k V^T$, where $\Sigma_k = \operatorname{diag}(\sigma_1, \sigma_2, \dots, \sigma_k, 0, \cdots, 0)$.

Then $\operatorname{rank}(A_k) = k$ and $\|A - A_k\|_2 = \sigma_{k+1}$

Furthermore, $\|A - A_k\|_2 = \cancel{\max} \min \{\|A - B\|_2 : \operatorname{rank}(B) \leq k\}$

So, A_k is the best approximation of rank less than or equal to k to A.
(However, many equally good (other) approximations in $\|\cdot\|_2$ exist)

Proof

$\|A - A_k\|_2 = \|U \operatorname{diag}(0, \cdots, 0, \sigma_{k+1}, \sigma_{k+2}, \cdots, \sigma_r, 0 \cdots 0) V^T\|_2 = \sigma_{k+1}$

A_k has rank k by construction.

Let B have rank k or less. Then $\dim(\operatorname{Null}(B)) \geq n - k$

Now we construct a vector x such that $\|(A-B)x\|_2 \geq \sigma_{k+1}$

If $\begin{cases} x \in \operatorname{span}\{v_1, v_2, \dots, v_{k+1}\} \\ \|x\|_2 = 1 \end{cases}$ then $\|Ax\|_2 \geq \sigma_{k+1}$.

Since $\dim(\operatorname{span}\{v_1, \dots, v_{k+1}\}) + \dim(\operatorname{Null}(B)) \geq n+1$, there is such a vector $x \in \operatorname{Null}(B)$. Take $x = \sum_{i=1}^{k+1} v_i \xi_i$ s.t. $Bx = 0$

Then $\|(A-B)x\|_2 = \|Ax\|_2 \geq \sigma_{k+1}$.

Hence, for any $B \in \mathbb{R}^{m\times n}$, such that $\operatorname{rank}(B) \leq k$, we have $\|A - B\|_2 \geq \sigma_{k+1}$.

$\|x\|_2 = 1$ and $x = v_1 \xi_1 + v_2 \xi_2 + \cdots + v_k \xi_k + v_{k+1} \xi_{k+1}$

$\rightarrow \sum_i \xi_i^2 = 1$

$$Ax = Av_1 \xi_1 + \cdots + Av_{k+1}\xi_{k+1} = u_1 \sigma_1 \xi_1 + u_2 \sigma_2 \xi_2 + \cdots + u_{k+1}\sigma_{k+1}\xi_{k+1}$$

$$\|Ax\|_2^2 = \sigma_1^2 \xi_1^2 + \sigma_2^2 \xi_2^2 + \cdots + \sigma_{k+1}^2 \xi_{k+1}^2$$

$$\geq \sigma_{k+1}^2 \cancel{\xi_{k+1}} \xi_1^2 + \sigma_{k+1}^2 \xi_2^2 + \cdots + \sigma_{k+1}^2 \xi_{k+1}^2$$

$$= \sigma_{k+1}^2 (\xi_1^2 + \cdots + \xi_{k+1}^2) = \sigma_{k+1}^2$$

$$\|Ax\|_2 \geq \sigma_{k+1}$$

Corollary 4.2.16

$A \in \mathbb{R}^{m \times n}$ has full rank $\operatorname{rank}(A) = r = \min(m,n)$

$\sigma(A) = \{\sigma_1, \sigma_2, \dots, \sigma_r\}$ where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$.

Let $B \in \mathbb{R}^{m \times n}$ and $\|A - B\|_2 < \sigma_r$. Then B has full ran

Let $A \in \mathbb{R}^{n \times n}$ be nonsingular. The distance ~~[illegible]~~ (measured by two norm) of A to nearest singular matrix is its smallest singular value.

$$\min \{ \|A - A_s\|_2 : A_s \in \mathbb{R}^{n \times n} \text{ is singular} \} = \sigma_n$$

where $\sigma(A) = \{\sigma_1, \dots, \sigma_n\}$ with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$

Solving least squares problems using the SVD

$Ax \simeq b$

$A \in \mathbb{R}^{m \times n}$, $m \geq n$, $A = U \Sigma V^T$, $\Sigma = \mathrm{diag}(\sigma_1, \cdots, \sigma_r, 0 \cdots 0)$

$\|b - Ax\|_2$ min $\qquad r \leq n$

Decompose b along U : $b = UU^T b = U\beta$

Decompose x along V : $x = VV^T x = V\xi$

$\|b - Ax\|_2 = \|U\beta - U \Sigma V^T V \xi\|_2 = \|\beta - \Sigma \xi\|_2$

$$\beta - \Sigma \xi = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{pmatrix} - \begin{pmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \\ \hline & & & O & & \end{pmatrix} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}$$

$$\|b - Ax\|_2 = \left\| \begin{pmatrix} \beta_1 - \sigma_1 \xi_1 \\ \vdots \\ \beta_r - \sigma_r \xi_r \\ \beta_{r+1} \\ \vdots \\ \beta_m \end{pmatrix} \right\|_2 \text{ min} \longrightarrow \begin{array}{l} \xi_i = \beta_i / \sigma_i \text{ for } i = 1..r \\ \xi_{r+1} .. \xi_n \text{ arbitrary} \end{array}$$

$$x = v_1 \beta_1/\sigma_1 + v_2 \frac{\beta_2}{\sigma_2} + \cdots + v_r \frac{\beta_r}{\sigma_r} + v_{r+1} \xi_{r+1} + \cdots + v_n \xi_n$$

Since components $v_{r+1}\xi_{r+1}, \cdots, v_n \xi_n$ arbitrary it makes sense to consider the (unique) minimum norm solution $x = \sum_{i=1}^{r} v_i \beta_i / \sigma_i$

$$A = U\Sigma V^T = \sum_{i=1}^{r} \sigma_i u_i v_i^T = U_r \Sigma_r V_r^T$$

$$U_r = [u_1 \cdots u_r], \quad \Sigma_r = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{pmatrix}, \quad V_r = [v_1 \cdots v_r]$$

Now the minimum norm solution x of $Ax \simeq b$ is given by $x = V_r \Sigma_r^{-1} U_r^T b$

The matrix $A^+ = V_r \Sigma_r^{-1} U_r^T$ is called the pseudo-inverse or Moore-Penrose generalized inverse (there are other generalized inverses).

$A \in \mathbb{R}^{m \times n}$, $m \geq n$: $A = U_r \Sigma_r V_r^T$

$A^+ \in \mathbb{R}^{n \times m}$, and $A^+A = V_r \Sigma_r^{-1} U_r^T U_r \Sigma_r V_r^T = V_r V_r^T$

$AA^+ = U_r U_r^T$

So, A^+A is orthog. projection onto Range(V_r)

AA^+ is orthog. projection onto Range(U_r)

if $r = n$ (A full column rank): $A^+A = I_n$

$(AA^+)A = A$

$(A^+A)A^+ = A^+$ and $A(A^+A) = A$

$(AA^+)^T = (U_r \Sigma_r V_r^T V_r \Sigma_r^{-1} U_r^T)^T = (U_r U_r^T)^T = U_r U_r^T = AA^+$

$(A^+A)^T = (V_r \Sigma_r^{-1} U_r^T U_r \Sigma_r V_r^T)^T = (V_r V_r^T)^T = V_r V_r^T = A^+A$

Sensitivity of LS

$A \in \mathbb{R}^{m\times n}, \quad b \in \mathbb{R}^m, \quad m \geq n$

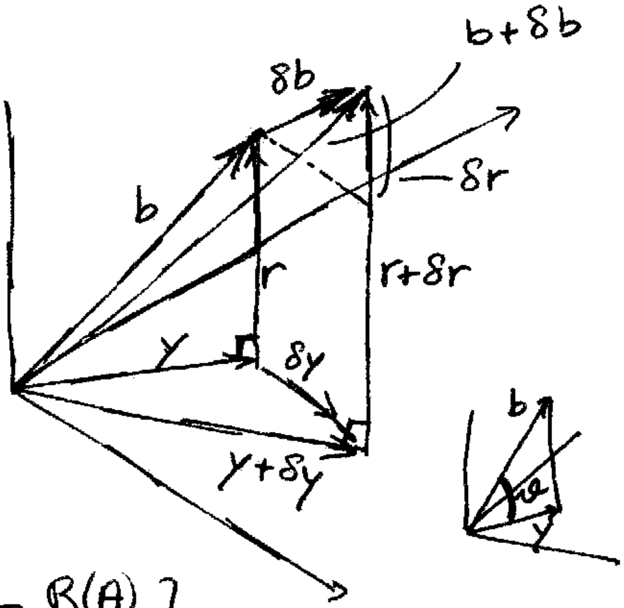
$Ax \simeq b \longrightarrow x = \arg\min_{\tilde{x}\in\mathbb{R}^n} \|b - A\tilde{x}\|_2$

Solution is two-step process:

(1) $y = \arg\min_{\tilde{y}\in R(A)} \|b-\tilde{y}\|_2 \quad , \quad r = b - y$

(2) Solve $Ax = y$ (exactly solvable since $y \in R(A)$)

Effect of perturbations on b



Perturbed right hand side $b+\delta b$

$(y+\delta y) = \arg\min_{\tilde{y}\in R(A)} \|b+\delta b - \tilde{y}\|_2$

Solve $A(x+\delta x) = y + \delta y \quad (\in R(A))$

$(b+\delta b) - (y+\delta y) = r + \delta r$

$$\left.\begin{array}{l} r+\delta r \perp R(A), \quad r \perp R(A) \implies \delta r \perp R(A) \\ y+\delta y \in R(A), \quad y \in R(A) \implies \delta y \in R(A) \end{array}\right\} \delta r \perp \delta y$$

$\delta b = \delta y + \delta r \longrightarrow \|\delta b\|_2^2 = \|\delta y\|_2^2 + \|\delta r\|_2^2$ and $\|\delta b\|_2 \geq \|\delta y\|_2$

$A(x+\delta x) = y + \delta y \longrightarrow A\,\delta x = \delta y$

$\|y\|_2 = \|b\|_2 \cos\theta \quad \left(\theta = \measuredangle\left(R(A), b\right)\right)$

LS sensitivity ~~of~~ wrt perturbations of right hand side

$$\|\delta x\|_2 = \|A^{-1}\delta y\|_2 \leq \|A^{-1}\|_2 \cdot \|\delta y\|_2 = \sigma_n^{-1}\|\delta y\|_2$$

$$\|y\|_2 = \|Ax\|_2 \leq \sigma_1\|x\|_2 \implies \|x\|_2 \geq \frac{\|y\|_2}{\sigma_1}$$

$$\frac{\|\delta x\|_2}{\|x\|_2} \leq \underbrace{\frac{\sigma_1}{\sigma_n} \cdot \frac{\|\delta y\|_2}{\|y\|_2}}_{\text{standard for solution of consistent linear system}} = \kappa_2(A) \cdot \frac{\|\delta y\|_2}{\|y\|_2} = \kappa_2(A) \underbrace{\frac{\|\delta y\|_2}{\|b\| \cos\vartheta}}_{\text{if } \vartheta \text{ near } \pi/2 \text{ then very ill-conditioned because } \|y\|_2 \text{ very small}}$$

$$\frac{\|\delta x\|_2}{\|x\|_2} \leq \frac{\kappa_2(A)}{\cos\vartheta} \frac{\|\delta y\|_2}{\|b\|_2} \leq \frac{\kappa_2(A)}{\cos\vartheta} \cdot \frac{\|\delta b\|_2}{\|b\|_2}$$

Sensitivity of LS for perturbations of A (and b)

$A \in \mathbb{R}^{m \times n}$, $m \geq n$, $\operatorname{rank}(A) = n$, $b \in \mathbb{R}^m$, $x \in \mathbb{R}^n$ (etc)

$$x = \arg\min_{\tilde{x}} \|b - A\tilde{x}\|_2$$

$$\angle(\mathcal{R}(A), b) = \vartheta < \pi/2$$

$$\varepsilon_A = \frac{\|\delta A\|_2}{\|A\|_2}, \quad \varepsilon_b = \frac{\|\delta b\|_2}{\|b\|_2}, \quad \varepsilon = \max(\varepsilon_A, \varepsilon_b) \ll 1$$

and $\varepsilon_A < \frac{1}{\kappa_2(A)} \longrightarrow \frac{\|\delta A\|_2}{\sigma_1} < \frac{\sigma_n}{\sigma_1} \Rightarrow \|\delta A\|_2 < \sigma_n$

So, $A + \delta A$ bounded away from singularity.

$$(x + \delta x) = \arg\min_{\tilde{x}} \|(b + \delta b) - (A + \delta A)\tilde{x}\|_2$$

Then $$\frac{\|\delta x\|_2}{\|x\|_2} \leq \frac{2\kappa_2(A)}{\cos\vartheta}\varepsilon_b + 2\left(\kappa_2(A)^2 \tan\vartheta + \kappa_2(A)\right)\varepsilon_A + O(\varepsilon^2)$$

Furthermore, let $r = b - Ax$ and $\hat{r} = r + \delta r$
$= (b + \delta b) - (A + \delta A)(x + \delta x)$

Then $$\frac{\|\delta r\|_2}{\|b\|_2} \leq 2\varepsilon_b + 3\kappa_2(A)\varepsilon_A + O(\varepsilon^2)$$

$A + \delta A$ nonsingular $\longrightarrow$

$$(A+\delta A)^T (A+\delta A)(x+\delta x) = (A+\delta A)^T (b+\delta b) \implies$$

$$\left(A^TA + A^T\delta A + \delta A^T A + \cancel{\delta A^T \delta A}\right)(x+\delta x) \cong A^Tb + A^T\delta b + \delta A^T b + \cancel{\delta A^T \delta b}$$

$$\cancel{A^TA\,x} + A^T\delta A x + \delta A^T A x + A^TA\,\delta x + \cancel{A^T\delta A \delta x} + \cancel{\delta A^T A \delta x} \cong \cancel{A^T b} + A^T\delta b + \delta A^T b \implies$$

$$\implies$$

$$\delta x \cong (A^TA)^{-1}\left(\delta A^T b - \delta A^T A x + A^T \delta b + A^T \delta A x\right) \implies$$

$$\delta x \cong (A^TA)^{-1}\delta A^T (b - Ax) + (A^TA)^{-1}A^T\delta b + (A^TA)^{-1}A^T\delta A\, x \implies$$

$$\|\delta x\|_2 \lesssim \sigma_n^{-2}\|\delta A\|_2 \|r\|_2 + \sigma_n^{-1}\|\delta b\|_2 + \sigma_n^{-1}\|\delta A\|_2 \|x\|_2 \implies$$

$$\|\delta x\|_2 \lesssim \kappa_2(A)^2 \frac{\|\delta A\|_2}{\|A\|_2}\cdot\frac{\|r\|_2}{\|A\|_2} + \kappa_2(A)\frac{\|\delta b\|_2}{\|A\|_2} + \kappa(A)\frac{\|\delta A\|_2}{\|A\|_2}\|x\|_2 \implies$$

$$\frac{\|\delta x\|_2}{\|x\|_2} \lesssim \kappa_2(A)^2 \varepsilon_A \frac{\|r\|_2}{\|A\|_2\|x\|_2} + \kappa_2(A)\frac{\|\delta b\|_2}{\|A\|_2\|x\|_2} + \kappa_2(A)\varepsilon_A \implies$$

$$\frac{\|\delta x\|_2}{\|x\|_2} \lesssim \kappa_2(A)^2 \varepsilon_A \frac{\|b\|\sin\theta}{\|b\|\cos\theta} + \kappa_2(A)\frac{\|\delta b\|_2}{\|b\|_2\cos\theta} + \kappa_2(A)\varepsilon_A \implies$$

$$\frac{\|\delta x\|_2}{\|x\|_2} \lesssim \kappa_2(A)^2\varepsilon_A \tan\theta + \kappa_2(A)\varepsilon_A + \frac{\kappa_2(A)}{\cos\theta}\varepsilon_b$$

$$\kappa_2(A)\varepsilon_A\left(\kappa_2(A)\tan\theta + 1\right) + \frac{\kappa_2(A)}{\cos\theta}\varepsilon_b$$