Intro Sys. Nonlinear Eq-s

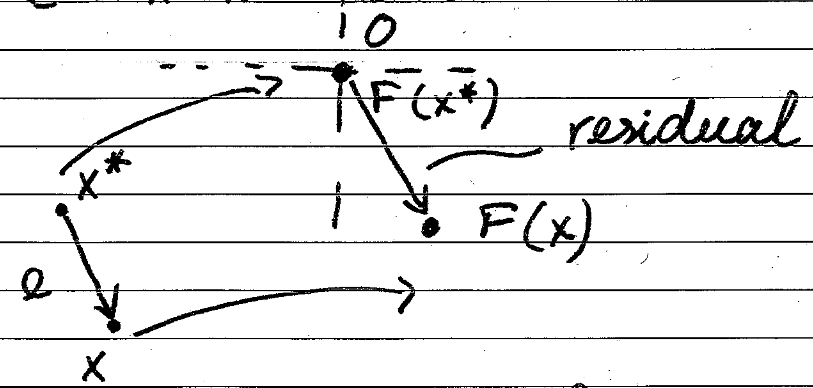
$F: \mathbb{R}^n \to \mathbb{R}^n$, $F(x) = 0$

$F(x)$ (esp. if x not solution) : (nonlinear) residual

x^* solution

Methods generate sequence $\{x_n\}_{n \geq 0}$ iterates

$e = x - x^*$, error



$$F(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix} \to f_i(x_1, x_2, \dots, x_n)$$

If all partial derivatives exist at x, we call matrix $F'(x) = J(x)$ with

$$F'(x)_{ij} = \frac{\partial f_i}{\partial (x)_j} = \frac{\partial (F)_i}{\partial (x)_j}(x) = J_{ij}(x)$$

the Jacobian (of F at x)

Given an iterate x_n we compute a new approximation using a 'local model' (typically linear)

$M_n(x) \simeq F(x)$ in neighborhood of x_n

This has 2 important consequences

1) We can solve the model equation (cheaply) but the solution will typically not be the exact solution of $F(x) = 0$ $\rightarrow$ model not exact

This also means we may not want to solve the model to high accuracy.

Furthermore, the accuracy (error) of next iterate will depend on the accuracy of the model, and the accuracy for which we solve model equation

2) The model is only locally valid

(and typically we have to estimate the "range" over which it is valid.)

Hence, if a step $x_{n+1} - x_n$ is too large, we cannot accept that step (in general), and we need additional modifications of the algorithm.

(line search, damping, trust region, ...)

The standard local model for Newton it.

$$M_n(x) = F(x_n) + F'(x_n)(x - x_n)$$

Exact Newton's method (Newton step) :

$$M_n(x_{n+1}) = 0 \Rightarrow F(x_n) + F'(x_n)(x_{n+1} - x_n) = 0 \Leftrightarrow$$

$$x_{n+1} - x_n = -\left(F'(x_n)\right)^{-1} F(x_n) \rightarrow$$

Newton : $x_{n+1} = x_n - \left(F'(x_n)\right)^{-1} F(x_n)$

Convergence theory for Newton's method (local)

local theory: x_0 "sufficiently" close to x^*

(we don't know how close this is)

Assumptions:

1) x^* exists
2) $F' : \Omega \to \mathbb{R}^{N \times N}$ Lipschitz continuous near x^*
3) $F'(x^*)$ is nonsingular

Lipschitz continuous near x^* $\rightarrow$

$$\| F'(x) - F'(y) \| \leq \gamma \|x - y\| \quad , \forall x, y \in N_{\varepsilon}(x^*) \; \left(B_\varepsilon(x^*)\right)$$

(define vector 2-norm + matrix induced norm)

<u>Theo:</u> (given Assumptions) If x_0 suff. close to x^*, then seq of Newton iterates exists and converges to x^* such that

$$\|e_{n+1}\| \leq K \|e_n\|^2$$

for some $K > 0$ and n sufficiently large.

<u>Proof:</u> (outline)

$$x_{n+1} = x_n - J(x_n)^{-1} F(x_n)$$

$$x_{n+1} - x^* = x_n - x^* - J(x_n)^{-1}\left(F(x_n) - F(x^*)\right)$$

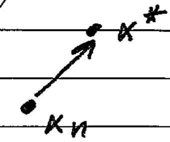
$$= J(x_n)^{-1}\left(F(x^*) - F(x_n) + J(x_n)(x_n - x^*)\right)$$

$$= J(x_n)^{-1}\left(F(x^*) - F(x_n) - J(x_n)(x^* - x_n)\right)$$

$$\left(\underline{\text{note}} \quad = J(x_n)^{-1}\left(F(x^*) - M_n(x^*)\right)\right)$$

$$\|e_{n+1}\| = \|x_{n+1} - x^*\| = \Big\| J(x_n)^{-1} \underbrace{\left(F(x^*) - F(x_n) - J(x_n)(x^* - x_n)\right)}_{\text{bound}} \Big\|$$

$$F(x^*) - F(x_n) - J(x_n)(x^* - x_n)$$



Consider $\hat{F}(s) = F(x_n + s(x^* - x_n))$

$$\hat{F}(1) = F(x^*) \qquad \hat{F}(0) = F(x_n)$$

Now components / coeff. of $\hat{F}(s)$ are $\hat{F}_i : \mathbb{R} \to \mathbb{R}$
(so std calculus)

$$\hat{F}_i(s) = F_i(x_n + s(x^* - x_n)) \rightarrow \frac{d}{ds}\hat{F}_i(s) = \sum_j \frac{\partial \hat{F}_i}{\partial (x)_j}(s) \cdot \frac{d}{ds}\left((x_n)_j + s((x^*)_j - (x_n)_j)\right)$$

$$\hat{F}_i(1) - \hat{F}_i(0) = \int_0^1 \frac{d}{ds}\hat{F}_i(s)\, ds$$

$$\hat{F}(1) - \hat{F}(0) = \int_0^1 J(x_n + s(x^* - x_n))(x^* - x_n)\, ds$$

$$\frac{d}{ds}\hat{F}_i(s) = \sum_j \frac{\partial \hat{F}_i}{\partial (x)_j}(s) \cdot ((x^*)_j - (x_n)_j)$$

$$= \sum_j \frac{\partial F_i}{\partial (x)_j}(x_n + s(x^* - x_n)) \cdot (x^* - x_n)_j$$

$$= \underbrace{\nabla F_i(x_n + s(x^* - x_n))^T}_{i\text{-th row of } J}(x^* - x_n)$$

$$F(x^*) - F(x_n) = \hat{F}(1) - \hat{F}(0) = \int_0^1 J(x_n + s(x^* - x_n))(x^* - x_n)\, ds$$

$$F(x^*) - F(x_n) - J(x_n)(x^* - x_n) =$$

$$\int_0^1 \left[J(x_n + s(x^* - x_n)) - J(x_n)\right](x^* - x_n)\, ds$$

$$\| F(x^*) - F(x_n) - J(x_n)(x^* - x_n) \| = \left\| \int_0^1 \left[J(x_n + s(x^* - x_n)) - J(x_n)\right](x^* - x_n)\, ds \right\|$$

$$\leq \int_0^1 \left\| \left[J(x_n + s(x^* - x_n)) - J(x_n)\right](x^* - x_n) \right\| ds$$

$$\leq \int_0^1 \gamma \|s(x^* - x_n)\| \|x^* - x_n\|\, ds \leq \gamma \|x^* - x_n\|^2 \int_0^1 s\, ds$$

$$= \frac{\gamma}{2}\|x^* - x_n\|^2 = \frac{\gamma}{2}\|e_n\|^2$$

$$\|e_{n+1}\| \leq \|J(x_n)^{-1}\| \cdot \|F(x^*) - F(x_n) - J(x_n)(x^* - x_n)\|$$
$$\leq \frac{\gamma \tilde{K}}{2} \|e_n\|^2$$

where $\tilde{K} \geq \|J(x)^{-1}\|$ over neighborhood of x^*

Based on this result, do proof by induction.

Issues:

a) We assume $\|J(x^*)^{-1}\| \leq \beta$ (for some $\beta > 0$).

based on this and Lipschitz property need to show that $\|J(x_n)^{-1}\|$ remains bounded over some region / neighborhood of x^*

b) We need to assure that all iterates x_n remain in neighborhood of x^* where $J = F'$ exists,

$\|J^{-1}(x_n)\|$ bounded, Lipschitz property holds.

c) We need to ensure a reduction of error, for example,

$\|e_{n+1}\| \leq \frac{1}{2}\|e_n\|$

(at some point $\|e_{n+1}\| \leq K\|e_n\|^2$ will be stronger)

d) We assume open convex set $D \subset \mathbb{R}^N$ such that F cont. diff. in D.
Take neighborhood $N(x^*, r) = \{\tilde{x} : \|x^* - \tilde{x}\| < r\} \subset D$

such that $J \in Lip_\gamma(N(x^*, r))$

e) We have a lot of freedom in choosing norm we use, but preferably use corr./induced matrix norm

This leads to requirement that $x_0 \in N(x^*, \varepsilon)$ with

$$\varepsilon = \min\left(r, \frac{1}{2\beta\gamma}\right)$$

To finish the proof we need the following results (proof in finite dimensional setting is not hard and is left for homework)

Let $E \in \mathbb{R}^{N \times N}$ and $\|E\| < 1$ (where $\|\cdot\|$ is an induced norm and hence consistent, also implies $\|I\| = 1$)

Then $(I - E)$ is invertible ($(I-E)^{-1}$ exists) and

$$\|(I-E)^{-1}\| \leq \frac{1}{1 - \|E\|}$$

Let A be invertible. Then if B is sufficiently near A, B is invertible too. More precisely

If $\|A^{-1}(B-A)\| < 1$, then

B^{-1} exists and $\|B^{-1}\| \leq \dfrac{\|A^{-1}\|}{1 - \|A^{-1}(B-A)\|}$

We use this to prove that for x sufficiently close to x^*

$J(x)^{-1}$ exists and we can bound $\|J(x)^{-1}\|$

There exists $\varepsilon > 0$ such that $\forall x \in N(x^*, \varepsilon)$:

$J(x)^{-1}$ exists and $\|J(x)^{-1}\| \leq 2\beta$

$(\beta = \|J(x^*)^{-1}\|)$

Consider $\|J(x^*)^{-1}(J(x) - J(x^*))\| \leq \overbrace{\|J(x^*)^{-1}\|}^{\beta} \cdot \|J(x) - J(x^*)\|$

$\leq \beta\gamma\|x - x^*\|$ (using Lipschitz cont.)

If x such that $\beta\gamma\|x - x^*\| < \frac{1}{2}$ then using the above

$$\|J(x)^{-1}\| \leq \frac{\|J(x^*)^{-1}\|}{1 - \|J(x^*)^{-1}(J(x) - J(x^*))\|} \leq \frac{\beta}{1 - 1/2} = 2\beta$$

So, taking $\gamma\beta \|x - x^*\| < \frac{1}{2} \iff \|x - x^*\| < \frac{1}{2\beta\gamma}$

gives $\|\mathcal{J}(x)^{-1}\| \leq 2\beta$

So, we can take $\varepsilon = \frac{1}{2\beta\gamma}$

Substitute this result in

$$\|e_{n+1}\| \leq \|\mathcal{J}(x_n)^{-1}\| \cdot \|F(x^*) - F(x_n) - \mathcal{J}(x_n)(x^* - x_n)\|$$

$$\leq 2\beta \cdot \frac{\gamma}{2}\|e_n\|^2$$

$$= \gamma\beta\|e_n\|^2$$

Now assume $x_0 \in N(x^*, \varepsilon)$. Then

$$\|e_0\| = \|x_0 - x^*\| < \varepsilon \implies$$

$$\|e_1\| \leq \gamma\beta\|e_0\| \cdot \|e_0\|$$

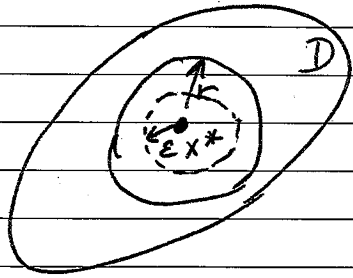
$$\leq \gamma\beta\varepsilon\|e_0\| = \frac{1}{2}\|e_0\|$$

$\left(\text{since } \varepsilon = \frac{1}{2\beta\gamma} \iff \beta\gamma\varepsilon = \frac{1}{2}\right)$

So, $x_1 \in N(x^*, \varepsilon)$ and we can prove convergence by induction (linear converge at least).

We assume F continuously differentiable in $N(x^*, r)$. We must take region where all conditions hold.

So, $x_0 \in N(x^*, \eta)$ where $\eta = \min(r, \frac{1}{2\beta\gamma})$



Convergence $\|e_{n+1}\| \le K \|e_n\|^2$ is called q-quadratic convergence

(overview of types of convergence)

→ number of significant (accurate) digits in answer roughly 'doubles' each iteration

(unfortunately, we don't know error)

→ if $F'(x^*)$ not ill-conditioned we ~~se~~ see quadratic reduction of ~~error~~ residual

$$F(x_{n+1}) = F(x^*) + J(x^*)(x_{n+1} - x^*)$$

$$\|F(x_{n+1})\| \le \|J(x^*)\| \, \|e_{n+1}\|$$

↳ bnd by constant but if $J(x^*)$ ill-cond then $\|J(x^*)\,p\|$ may depend strongly on 'direction' of p

Steps of Newton algorithm

1) evaluate $F(x_n)$ and test for convergence

2) approximate solution of $F'(x_n)s = -F(x_n)$

or more generally $M_n(\tilde{x}) = 0$ or $\|M_n(\tilde{x})\| \leq tol$

(approx. solution local model)

3) $x_{n+1} = x_n + \lambda s$, where steplength is selected to guarantee (sufficient) decrease of $\|F\|$

(in addition step should ~~~~ not be too small)

Typically (2) is expensive part, but (3) can be expensive if function eval is expensive (and needs to be done multiple times)

Show 1D examples

1) $\arctan(x) = 0$

2) $\tan(x) - x = 0$

discuss:

a) Need line search when far from solution, more general issue of ill-cond. Jac (scalar $F' \simeq 0$) and limitations of local model. (stepsize control)

ill-cond $\rightarrow$ directions in which in rel. sense F changes very slowly $\rightarrow$ geometric notions

b) Accuracy we can obtain bounded by accuracy in function evaluation and accuracy in derivative evaluation

note Newton step is "residual correction"

$$s_n = J(x_n)^{-1}(F(x_n)+\varepsilon_n)$$

$$\tilde{x}_{n+1} = x_n - s_n = x_n - J_n^{-1}F_n - J_n^{-1}\varepsilon_n$$

bnd on acc. in solution based on $\|J(x)^{-1}\|\,\|\varepsilon_n\|$

will also show up in convergence of $\|F_n\|$

(not on acc. Jac. if $\|\Delta\| \le C\|e_n\|$)

In general acc. in $F(x)$ not issue; conv. will stagnate at some point, but typ. suff. accurate.

We may not want to compute exact $F'(x_n)$ (cost)

We may not want to solve model eq. exactly (even with approx. F')

Convergence result for Newton with inexact F' and inexact F.

Theorem 1.2 (Std. Assumptions)

matrix $\Delta(x)$: $\|\Delta(x)\| < \delta_J$
vector $\varepsilon(x)$: $\|\varepsilon(x)\| < \delta_F$

for all x near x^* (neighborhood)

If x_0 suff. near x^* and δ_J, δ_F suff. small then

$$x_{n+1} = x_n - \left(F'(x_n)+\Delta(x_n)\right)^{-1}\left(F(x_n)+\varepsilon(x_n)\right)$$

well-defined ($(F'(x_n)+\Delta(x_n))^{-1}$ exists (bounded))

and

$$\|e_{n+1}\| \le \bar{K}\left(\|e_n\|^2 + \|\Delta(x_n)\|\|e_n\| + \|\varepsilon(x_n)\|\right)$$

for some $\bar{K} > 0$

Proof see Kelley, Iterative Methods for Linear and Nonlinear Equations (IMLNE)

(proof follows same lines as "exact" version with some extra bounds)

Note that errors in Jac. eval. lead, in general, to linear convergence.

If $\|\Delta(x_n)\| \le c\|e_n\|$ and $\varepsilon(x_n)$ suff. small, we retain quadratic convergence.

As Newton's method can be expensive we often look for ways to reduce cost.

Main costs (<u>usually</u>) : i) compute $F'(x_n)$ (N^2)
ii) solve for $F'(x_n)$ (N^3)

→ Approximate Jacobian,

- * especially efficient if approx. also reduces cost of solving for Jacobian
- * balance slower convergence with (significantly) lower cost per iteration.

1) Chord method / modified Newton

replace $F'(x_n)$ by $F'(x_0)$ (compute once, factorise once)

If x_0 close enough to x^* then convergence is q-linear : $\|e_{n+1}\| \le \rho\|e_n\|$ $(\rho \in [0,1))$ for n suff. large

Using Theo 1.2 : $\left(F'(x_n) + F'(x_0) - F'(x_n)\right)^{-1} \dots$

so $\Delta(x_n) = F'(x_0) - F'(x_n) \Rightarrow$

$$\|\Delta(x_n)\| = \|F'(x_0) - F'(x_n)\| \leq \gamma \|x_0 - x_n\|$$

Assuming criteria satisfied and hence convergence (strict monotone → reduction each step)

$$\|\Delta(x_n)\| \leq \gamma \|x_0 - x_n\| = O(\|e_0\|)$$

If we take $N_r(x^*)$ where convergence guaranteed and (obviously $x_0 \in N_r(x^*)$) $\Rightarrow$

$$\|\Delta(x_n)\| \leq \gamma \|x_0 - x_n\| \leq 2\gamma \|x_0 - x^*\| = 2\gamma \|e_0\|$$

Taking $\varepsilon(x_n) = 0$ or suff. small (usually machine precision)
↳ negligible

$$\|e_{n+1}\| \leq \bar{K}\left(\|e_n\|^2 + C\|e_0\| \cdot \|e_n\| + \varepsilon\right)$$

$$\leq \rho \|e_n\|$$ if $\|e_n\|$ suff. small and $\|e_0\|$ → ρ

ρ is called q-factor

For q-linear convergence typically

$$\|e_{n+1}\| \simeq \rho \|e_n\| \text{ and } \|F(x_{n+1})\| \simeq \rho \|F(x_n)\|$$

2) Secant method

1D : $x_{n+1} = x_n - F(x_n) \Big/ \dfrac{F(x_n) - F(x_{n-1})}{x_n - x_{n-1}} \iff$

$$x_{n+1} = x_n - \frac{F(x_n)(x_n - x_{n-1})}{F(x_n) - F(x_{n-1})}$$

backward difference approx. for $F'(x_n) \simeq \dfrac{F(x_n) - F(x_{n-1})}{x_n - x_{n-1}}$

$$F(x_n) - F(x_{n-1}) = \int_{x_{n-1}}^{x_n} F'(s)\,ds$$

$$\left| \int_{x_{n-1}}^{x_n} F'(s)\,ds - F'(m)(x_n - x_{n-1}) \right| = \left| \int_{x_{n-1}}^{x_n} F'(s) - F'(m)\,ds \right|$$

$$\leq \int_{x_{n-1}}^{x_n} |F'(s) - F'(m)|\,ds \leq \int \gamma |s-m|\,ds = \tfrac{1}{2}\int_{x_{n-1}}^{m} (m-s)\gamma\,ds + \tfrac{1}{2}\int_{m}^{x_n} \gamma(s-m)\,ds$$

$$= \frac{\gamma}{4}(x_n - x_{n-1})^2$$

or Taylor $F(x_{n-1}) = F(x_n) + F'(\xi)(x_{n-1} - x_n) \quad \xi \in (x_{n-1}, x_n)$

$$x_{n+1} = x_n - \frac{F(x_n)\cancel{(x_n - x_{n-1})}}{F'(\xi)\cancel{(x_n - x_{n-1})}} \quad \xi \in (x_{n-1}, x_n) \Rightarrow \|\Delta(x_n)\| \leq \|e_{n-1}\|$$

$$|e_{n+1}| \leq \bar{K}\left(|e_n|^2 + |e_{n-1}||e_n| + \varepsilon\right) \Rightarrow$$

$$\frac{|e_{n+1}|}{|e_n|} \leq \bar{K}\left(|e_n| + |e_{n-1}|\right) \leq 2\bar{K}|e_{n-1}| \rightarrow 0$$

q-superlinearly convergent

There is no "obvious" extension to higher dimensions but many methods have been proposed.

Most well-known: Broyden's method (more than one)

→ BFGS, ...

Basic idea: update (approximate) Jacobian by low rank update satisfying certain conditions

Another way to reduce costs are

Inexact Newton methods $\rightarrow$ solve Newton step inexactly

Replace $F'(x_n)s = -F(x_n)$

by $\|F'(x_n)s + F(x_n)\| \le \eta \|F(x_n)\|$

where $s = x_{n+1} - x_n$ (Newton step)

Forcing term η typically varied during iteration

η small $\rightarrow$ close to Newton but expensive
η large $\rightarrow$ slower convergence " cheaper

Far from solution Newton converges linear at best
$\rightarrow$

sequence $\eta_n \rightarrow 0$

Cheap far from solution
Recover (nearly) quadratic convergence close to solution

<u>Theo 1.3</u> (Std Assumptions)

There are δ and $\bar{\eta}$ such that, if $x_0 \in B(\delta)$,

$\{\eta_n\} \subset [0, \bar{\eta}]$, then inexact Newton iteration

$x_{n+1} = x_n + s_n$, where

$\|F'(x_n)s_n + F(x_n)\| \le \eta_n \|F(x_n)\|$

converges q-linearly to x^*

Moreover,

* if $\eta_n \to 0$, convergence is q-superlinear
* if $\eta_n \leq K_\eta \|F(x_n)\|^p$ for some $K_\eta > 0$, convergence is q-superlinear with q-order $1+p$

<u>Proof</u> (take $\Delta(x_n) = 0$) $\rightarrow$ second part

$$F'(x_n)\, s_n = \underbrace{-F(x_n) + \varepsilon(x_n)}_{} \quad \text{where } \|\varepsilon(x_n)\| \leq \eta_n \|F(x_n)\|$$

↳ corresponds to inexact function eval plus exact solve

$$F(x_n) = F(x^*) + \int_0^1 F'(x^* + t(x_n - x^*))(x_n - x^*)\, dt$$

$$= \underset{0}{F(x^*)} + F'(x^*)(x_n - x^*) + \tilde{\varepsilon} \qquad \|\tilde{\varepsilon}\| \leq \frac{\gamma}{2}\|e_n\|^2$$

$$F(x_n) = F'(x^*)(x_n - x^*) + \tilde{\varepsilon}$$

$$\|F(x_n)\| \leq \|F'(x^*)\| \cdot \|e_n\| + \gamma/2 \, \|e_n\|^2$$

$$= \|e_n\| \left(\|F'(x^*)\| + \gamma/2 \, \|e_n\| \right)$$

Theo 1.2 $\rightarrow \|e_{n+1}\| \leq \bar{K} \left(\|e_n\|^2 + \eta_n \|e_n\| \left(\|F'(x^*)\| + \gamma/2 \, \|e_n\| \right) \right)$

$$\eta_n \leq K_\eta \|F(x_n)\|^p \leq K_\eta \|e_n\|^p \left(\|F'(x^*)\| + \gamma/2 \, \|e_n\| \right)^p$$

$$\|e_{n+1}\| \leq \bar{K} \Big(\|e_n\|^2 + \underbrace{K_\eta \|e_n\|^{1+p} \big(\underbrace{\|F'(x^*)\| + \gamma/2 \, \|e_n\|}_{\geq \|F'(x^*)\|} \big)^{1+p}}_{\tilde{K}} \Big)$$

convergence is q-superlinear with q-order $1+p \leq 2$

Termination Criterion: $\|F(x)\| \leq \tau_r \|F(x_0)\| + \tau_a$

(as before) $N(x^*, \varepsilon)$ such that $\quad B(\varepsilon)$ in book

$$x \in N(x^*, \varepsilon) : \|F'(x)\| \leq 2\|F'(x^*)\| \quad \text{and}$$

$$\|(F'(x))^{-1}\| \leq 2\|(F'(x^*))^{-1}\|$$

Then $\|F'(x^*)^{-1}\|^{-1}\|e\|/2 \leq \|F(x)\| \leq 2\|F'(x^*)\| \cdot \|e\|$

Also holds for $F(x_0) \rightarrow$

$$\frac{\frac{1}{2}\|F'(x^*)^{-1}\|^{-1}\|e\|}{2\|F(x^*)\|\,\|e_0\|} \leq \frac{\|F(x)\|}{\|F(x_0)\|} \leq \frac{2\|F(x^*)\| \cdot \|e\|}{\frac{1}{2}\|F(x^*)^{-1}\|^{-1} \cdot \|e_0\|} \iff$$

$$\frac{1}{4\kappa(F'(x^*))} \cdot \frac{\|e\|}{\|e_0\|} \leq \frac{\|F(x)\|}{\|F(x_0)\|} \leq 4\kappa(F(x^*)^{-1}) \cdot \frac{\|e\|}{\|e_0\|}$$

relative reduction of residual norm bounded from above and below by constant times relative reduction of error norm.

If superlinear convergence $\left(\frac{\|e_{n+1}\|}{\|e_n\|} \rightarrow 0\right)$

$$e_{n+1} = e_n + s_n \quad \leftrightarrow x_{n+1} - x^* = x_n - x^* + s_n$$
$$\|e_{n+1}\| = o(\|e_n\|)$$

Hence $s_n = -e_n + o(\|e_n\|) \Rightarrow \|s_n\| \simeq \|e_n\|$

(current) rate of convergence: $\rho_n = \frac{\|s_n\|}{\|s_{n-1}\|} \simeq \frac{\|e_n\|}{\|e_{n-1}\|} \geq \frac{\|e_{n+1}\|}{\|e_n\|}$

(for n suff. large)

So, for n suff. large:

$$\|e_{n+1}\| \leq \rho_n \|e_n\| \simeq \frac{\|s_n\|^2}{\|s_{n-1}\|}$$

For superlin conv. iteration

(stopping crit) $\|s_n\|^2 / \|s_{n-1}\| < \tau$

implies that $\|e_{n+1}\| < \tau$

What for linearly convergent process

$$\rho \simeq \frac{\|s_n\|}{\|s_{n-1}\|} \quad \text{or} \quad \rho \simeq \left(\frac{\|s_n\|}{\|s_0\|}\right)^{\frac{1}{n}}$$

$$e_n = e_{n+1} - s_n \Rightarrow \|e_n\| \le \|e_{n+1}\| + \|s_n\| \Rightarrow$$

$$\|e_n\| - \|s_n\| \le \|e_{n+1}\| \simeq \rho \|e_n\| \quad (*1)$$

$$\|e_{n+1}\|/\rho \simeq \|e_n\| \le \frac{\|s_n\|}{1-\rho}$$

Terminate when $\|s_n\| \le \frac{\tau(1-\rho)}{\rho}$, then if ρ is overestimate

$$\|e_{n+1}\| \le \rho \|s_n\|/(1-\rho) \le \tau$$

(in practice use an additional safety factor)

$$(*1) \quad -\|s_n\| \lesssim \|e_n\|\rho - \|e_n\| = \|e_n\|(\rho - 1)$$

$$\|s_n\| \gtrsim \|e_n\|(1-\rho)$$

$$\|e_n\| \lesssim \|s_n\| \frac{1}{1-\rho}$$

$$\|e_{n+1}\| \simeq \rho\|e_n\| \lesssim \|s_n\| \frac{\rho}{1-\rho}$$

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Line Search / Armijo Rule

If local model violated → Newton step too large, we cannot trust Newton ~~sys~~ step (solution) to be accurate, or even F to be defined.

Extreme case $F'(x_n)$ is singular $\quad \|F(x_n)'^{-1}\| = \infty$

Line search (damped Newton) to reduce step size

$d_n = -(F'(x_n))^{-1} F(x_n) \quad , \quad s = \lambda d_n$

1D equation $L(\lambda) = \| F(x_n + \lambda d_n) \|^2$

Sufficient decrease (Armijo Rule):

$\| F(x_n + 2^{-m} d_n) \| < (1 - \alpha 2^{-m}) \| F(x_n) \| \quad$ for $\alpha \in (0,1)$

typical value $\alpha = 10^{-4}$

accept smallest $m \geq 0$ that satisfies rule

In some cases not aggressive enough, many line search steps → too expensive
eval. of $F(x_n + \lambda d_n)$ can be quite expensive

Note that for $m = 1$ we have 3 values for $L(\lambda)$:

$L(0), L(1/2), L(1)$ → use for poly. approx. and minimize polynomial (cheap!) subject to reasonable decrease of stepsize: $p(\lambda)$ fits $L(0), L(\lambda_m), L(\lambda_{m-1})$; find

$\min\{ p(\lambda) : \lambda \in [\frac{\lambda_m}{10}, \frac{\lambda_m}{2}]$

Linesearch only robust if $F'(x_n)$ accurate. So, if linesearch not effective and using approx. Jac. → compute (more) accurate Jacobian.

Linesearch ends for smallest $m \geq 0$ such that

$$\| F(x_n + \lambda_m d) \| < (1 - \alpha \lambda_m) \| F(x_n) \|$$

Basic Newton Algorithm

Input x (initial guess)
F function handle / pointer
τ_a absolute tolerance
τ_r relative tolerance

Eval. $F(x)$; $\tau = \tau_r \| F(x) \| + \tau_a$

while $\| F(x) \| > \tau$ do
"Solve" for d s.t. $\| F(x) + F'(x) d \| \leq \eta \| F(x) \|$
(terminate with failure if not successful)

$\lambda = 1$
while $\| F(x + \lambda d) \| \geq (1 - \alpha \lambda) \| F(x) \|$ do

$\lambda = \sigma \lambda$, where $\sigma \in [\frac{1}{10}, \frac{1}{2}]$ minimizing polynomial model of $\| F(x + \lambda d) \|^2$

end while
$x = x + \lambda d$
end while

One of 3 possibilities:
i) $\{x_n\} \to x^*$ where std. assump. hold
ii) $\{x_n\}$ is unbounded
iii) $\{F'(x_n)\}$ will become singular

See Theo: 1.4

Read remainder of chapter (1.7.1 ff)