Iterative Methods and Multigrid

Part 4: Local mode analysis

Local mode analysis

How to analyze smoothing behavior for more general problems and relaxations: local mode (normal mode/Fourier) analysis.
In general computing eigenvectors/values is too hard (harder than solvng a linear system)

The idea is to derive the smoothing factor from idealization of the equations at a point (experiment to find worst point).

This separates the smoothing of the error from the other algorithmic components. It also provides a optimal figure against which to compare overall performance of algorithm.

Idealizations:
- Assume infinite domain
- Assume equations are same everywhere
- Assume relaxation scheme is linear process

Local mode analysis

Error given by linear iteration: $e^{(k+1)} = Ge^{(k)}$

Now we assume the error consists of Fourier modes and we analyze how relaxation acts on these modes (ideally no mixing of modes). We should analyze the damping of the eigenvectors of the iteration matrix. However, we assume that oscillating modes are approximately eigenvectors (generally true); and apply smoothing analysis to these.

For discrete domain we have waves $w_j = \sin\frac{jk\pi}{n}$ with wavenumber $k = 1 \ldots n$.
Now we replace $\frac{k\pi}{n}$ by θ (continuous wavenumber) and consider waves $w_j = \exp(\imath j\theta)$ with $\theta \in (-\pi, \pi]$.

Values $|\theta|$ near 0 correspond to low frequency waves; values of $|\theta|$ near π correspond to high frequency waves.

Wavelength of mode θ is $\frac{2\pi h}{|\theta|}$.

Local mode analysis

Error at point j at iteration k : $e_j^{(k)} = A(k)\exp(\imath j\theta),\ -\pi < \theta \leq \pi.$

Goal is to find relation $A(k+1) = A(k)G(\theta)$,
where $G(\theta)$ is called the amplification factor (for mode θ)

This way we can analyze the convergence of the modes separately.
For convergence (of relaxation method) we need $|G(\theta)| < 1$ for all θ.

For MG only need damping of oscillatory modes: $|G(\theta)| < 1,\ \frac{\pi}{2} \leq |\theta| \leq \pi$.

We define smoothing factor as $\mu = \max_{\frac{\pi}{2} \leq |\theta| \leq \pi} |G(\pi)|$.
(slowest damping of oscillatory modes)

1D Example: Jacobi

Consider one-dimensional equation $-u_{xx} + cu = f$

Central finite differences gives: $-v_{j-1} + (2 + h^2 c_j) v_j - v_{j+1} = h^2 f_j$

Jacobi: $v_j^{(k+1)} = \frac{1}{(2+h^2 c_j)} \left(h^2 f_j + v_{j-1}^{(k)} + v_{j+1}^{(k)} \right)$

weighted Jacoby: $v_j^{(k+1)} = \frac{\omega}{(2+h^2 c_j)} \left(h^2 f_j + v_{j-1}^{(k)} + v_{j+1}^{(k)} \right) + (1-\omega) v_j^{(k)}$

For the error we get:

$$e_j^{(k+1)} = u_j - v_j^{(k+1)} = \omega u_j - \frac{\omega}{(2+h^2 c_j)} \left(h^2 f_j + v_{j-1}^{(k)} + v_{j+1}^{(k)} \right) + (1-\omega) u_j - (1-\omega) v_j^{(k)}$$

Since u satisfies the equation we have $u_j = \frac{1}{(2+h^2 c_j)} \left(h^2 f_j + u_{j-1} + u_{j+1} \right)$

1D Example: Jacobi

Substituting $u_j = \frac{1}{(2+h^2 c_j)}(h^2 f_j + u_{j-1} + u_{j+1})$ in expression for error

$$e_j^{(k+1)} = \omega u_j - \frac{\omega}{(2+h^2 c_j)}\left(h^2 f_j + v_{j-1}^{(k)} + v_{j+1}^{(k)}\right) + (1-\omega)u_j - (1-\omega)v_j^{(k)}$$

gives

$$e_j^{(k+1)} = \frac{\omega}{(2+h^2 c_j)}(h^2 f_j + u_{j-1} + u_{j+1}) - \frac{\omega}{(2+h^2 c_j)}\left(h^2 f_j + v_{j-1}^{(k)} + v_{j+1}^{(k)}\right) + (1-\omega)e_j^{(k)}$$

and finally

$$e_j^{(k+1)} = \frac{\omega}{(2+h^2 c_j)}\left(e_{j-1}^{(k)} + e_{j+1}^{(k)}\right) + (1-\omega)e_j^{(k)}$$

1D Example: Jacobi

Now substitute $A(k)\exp(\imath j\theta)$ for $e_j^{(k)}$ and simplify expression
First we assume $c = 0$.

$e_j^{(k+1)} = \frac{\omega}{2}\left(e_{j-1}^{(k)} + e_{j+1}^{(k)}\right) + (1-\omega)e_j^{(k)}$ becomes

$$A(k+1)\exp(\imath j\theta) =$$
$$\frac{\omega}{2}(A(k)\exp(\imath(j-1)\theta) + A(k)\exp(\imath(j+1)\theta)) + (1-\omega)A(k)\exp(\imath j\theta)$$

$$A(k)\exp(\imath[j-1]\theta) + A(k)\exp(\imath[j+1]\theta) = A(k)\exp(\imath j\theta)[\exp(-\imath\theta) + \exp(\imath\theta)] =$$
$$A(k)\exp(\imath j\theta)[\cos\theta - \imath\sin\theta + \cos\theta + \imath\sin\theta] =$$
$$A(k)2\exp(\imath j\theta)\cos\theta$$

$$A(k+1)\exp(\imath j\theta) = \frac{\omega}{2}A(k)2\exp(\imath j\theta)\cos\theta + (1-\omega)A(k)\exp(\imath j\theta) =$$
$$A(k)\exp(\imath j\theta)[1-\omega(1-\cos\theta)]$$

1D Example: Jacobi

Using $\cos\theta = \cos 2\cdot(\frac{\theta}{2}) = 1 - 2\sin^2(\frac{\theta}{2})$ we get

$$A(k)\exp(\imath j\theta)[1-\omega(1-\cos\theta)] = A(k)\exp(\imath j\theta)[1-2\omega\sin^2(\tfrac{\theta}{2})]$$

This gives the amplification factor

$$A(k+1) = A(k)[1-2\omega\sin^2(\tfrac{\theta}{2})] = A(k)G(\theta)$$, for $-\pi < \theta \leq \pi$.

These are the same convergence rates we saw for discrete wavenumbers k, corresponding to $\theta_k = \frac{k\pi}{n}$.
This is not generally the case, however.
From the above result we know that optimal weight is $\omega = \frac{2}{3}$, which yields

$$\mu = G(\tfrac{\pi}{2}) = |G(\pm\theta)| = \tfrac{1}{3}$$

1D Example: Jacobi

Now we assume $c \neq 0$.

$e_j^{(k+1)} = \frac{\omega}{2+h^2 c_j}\left(e_{j-1}^{(k)} + e_{j+1}^{(k)}\right) + (1-\omega)e_j^{(k)}$ becomes

$$A(k+1)\exp(\imath j\theta) = \frac{\omega}{2+h^2 c_j} A(k) 2\exp(\imath j\theta)\cos\theta + (1-\omega)A(k)\exp(\imath j\theta) =$$
$$A(k)\exp(\imath j\theta)\left[1-\omega+\frac{2\omega\cos\theta}{2+h^2 c_j}\right] =$$
$$A(k)\exp(\imath j\theta)\left[1-\omega\left(1-\frac{2\cos\theta}{2+h^2 c_j}\right)\right]$$

Result depends on c_j. Typically take some fixed c for analysis: maximum or minimum c_j or worst case for amplification factor.

This gives for the amplification factor: $G(\theta) = 1-\omega\left(1-\frac{2\cos\theta}{2+h^2 c}\right)$ this can be rewritten (for comparison) as

$$G(\theta) = 1-\omega(1-\cos\theta)+\omega(1-\cos\theta)-\omega\left(1-\frac{2\cos\theta}{2+h^2 c}\right)$$

1D Example: Jacobi

The amplification factor $G(\theta) = 1 - \omega\left(1 - \frac{2\cos\theta}{2+h^2c}\right)$ can be rewritten (for comparison) as

$$
\begin{aligned}
G(\theta) = 1 - \omega(1-\cos\theta) + \omega(1-\cos\theta) - \omega\left(1 - \frac{2\cos\theta}{2+h^2c}\right) &= \\
G_0(\theta) + \omega\left(1 - \cos\theta - 1 + \frac{2\cos\theta}{2+h^2c}\right) &= \\
G_0(\theta) + \omega\left(-1 + \frac{2}{2+h^2c}\right)\cos\theta &= \\
G_0(\theta) - \frac{\omega h^2 c}{2+h^2c}\cos\theta &
\end{aligned}
$$

where $G_0(\theta)$ is $G(\theta)$ for the case $c = 0$.

$G(\theta)$ only differs significantly from $G_0(\theta)$ if h^2c is not too small.

1D Example: Gauss-Seidel

For Gauss-Seidel we get

$$e_j^{(k+1)} = \frac{e_{j-1}^{(k+1)} + e_{j+1}^{(k)}}{2+h^2 c_j},$$

assuming carry out the relaxations from left to right (increasing j).

Again assuming $e_j^{(k)} = A(k)\exp(\imath j\theta)$ and $c = 0$ we get

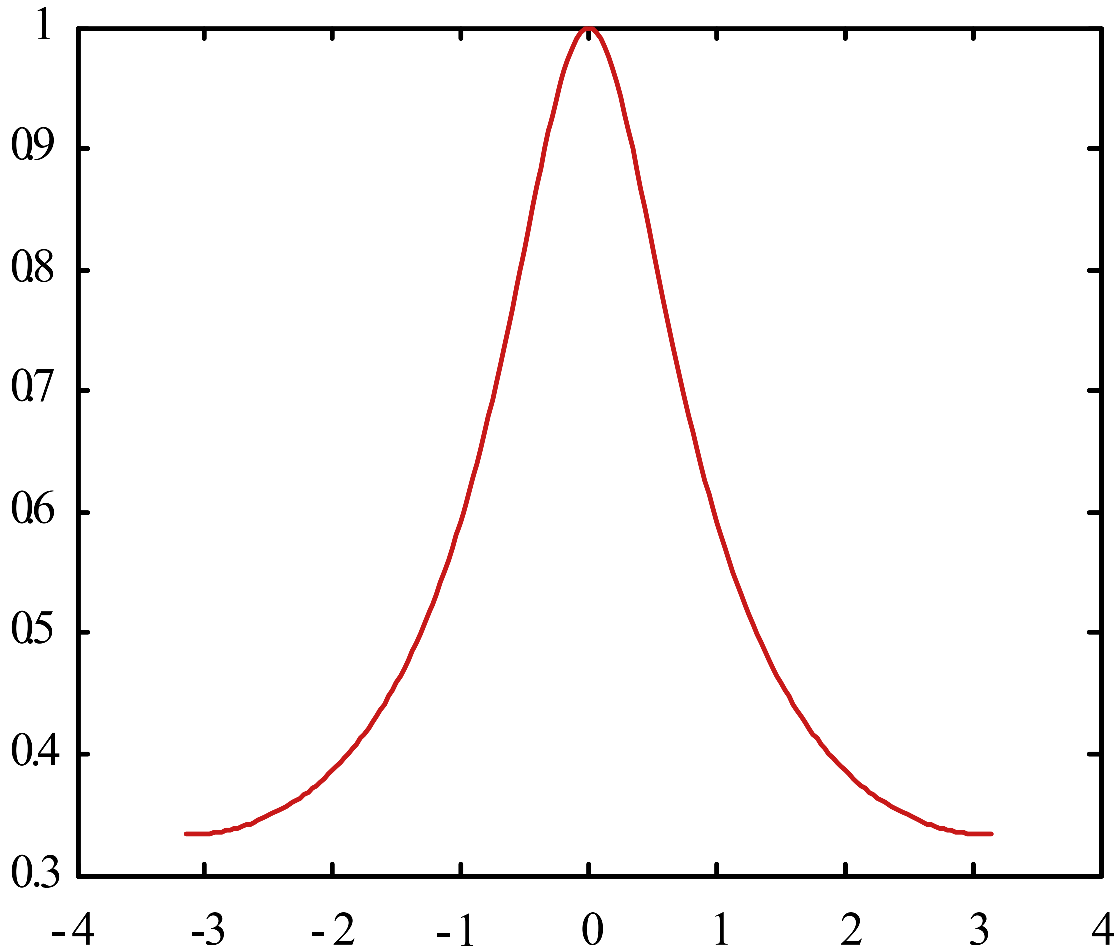
$$A(k+1) = \frac{\exp(\imath\theta)}{2-\exp(-\imath\theta)} A(k).$$

So we have as amplification factor $G(\theta) = \frac{\exp(\imath\theta)}{2-\exp(-\imath\theta)}$

The graph of $|G(\theta)|$ shows the smoothing factor is obtained for $\theta = \frac{\pi}{2}$.

$$\mu = |G(\tfrac{\pi}{2})| = \left|\frac{i}{2+i}\right| = \sqrt{\frac{1}{5}} \approx 0.453$$

1D Example: Gauss-Seidel



1D Example: Gauss-Seidel

Contrary to what we saw for the Jacobi iteration, we notice that in this case the eigenvalues on the infinite domain without boundaries are different from the case with boundaries.

So the result only approximately gives the smoothing factors on finite domains. As we can verify from the results from chapter 2.

Local mode analysis for 2D

We can apply local mode analysis analogously to two-dimensional problems.

We represent the error as $e_{jm}^{(k)} = A(k)\exp(ij\theta_1 + im\theta_2)$

where j and m run in x and y direction, and θ_1 and θ_2 represent waves in x and y direction.

Now we look for recurrence: $A(k+1) = G(\theta_1, \theta_2)A(k)$ where again

$G(\theta_1, \theta_2)$ is amplification factor corresponding to two wavenumbers.

The oscillatory modes are modes that are oscillatory in either one direction or both.

2D Example: Jacobi

Consider $-u_{xx} - u_{yy} + cu = f$

Discretization: $-u_{jm-1} - u_{j-1m} + (4 + h^2 c_{jm})u_{jm} - u_{j+1m} - u_{jm+1} = h^2 f_{jm}$

So weighted Jacobi iteration:

$$v_{jm}^{(k+1)} = \frac{\omega}{(4+h^2 c_{jm})}\left(h^2 f_{jm} + v_{jm-1}^{(k)} + v_{j-1m}^{(k)} + v_{j+1m}^{(k)} + v_{jm+1}^{(k)}\right) + (1-\omega)v_{jm}^{(k)}$$

For error this gives

$$e_{jm}^{(k+1)} = \frac{\omega}{(4+h^2 c_{jm})}\left(e_{jm-1}^{(k)} + e_{j-1m}^{(k)} + e_{j+1m}^{(k)} + e_{jm+1}^{(k)}\right) + (1-\omega)e_{jm}^{(k)}$$

For $c = 0$

$$e_{jm}^{(k+1)} = \frac{\omega}{4}\left(e_{jm-1}^{(k)} + e_{j-1m}^{(k)} + e_{j+1m}^{(k)} + e_{jm+1}^{(k)}\right) + (1-\omega)e_{jm}^{(k)}$$

2D Example: Jacobi

Analyze case $c = 0$

$$e_{jm}^{(k+1)} = \tfrac{\omega}{4}\left(e_{jm-1}^{(k)} + e_{j-1m}^{(k)} + e_{j+1m}^{(k)} + e_{jm+1}^{(k)}\right) + (1-\omega)e_{jm}^{(k)}$$

$$e_{jm}^{(k+1)} = \tfrac{\omega}{4}A(k)[\exp\imath[j\theta_1 + (m-1)\theta_2] + \exp\imath[j\theta_1 + (m+1)\theta_2] + \cdots] + (1-\omega)A..$$

$$\tfrac{\omega}{4}A(k)\exp\imath[j\theta_1 + m\theta_2](2\cos\theta_2 + 2\cos\theta_1) + (1-\omega)A(k)\exp\imath[j\theta_1 + m\theta_2] =$$

$$A(k)\exp\imath[j\theta_1 + m\theta_2](1-\omega[1-\tfrac{1}{4}(2\cos\theta_2 + 2\cos\theta_1)])$$

$$2\cos\theta_1 + 2\cos\theta_2 = 2\left(1-2\sin^2\left(\tfrac{\theta_1}{2}\right)\right) + 2\left(1-2\sin^2\left(\tfrac{\theta_2}{2}\right)\right) =$$

$$4 - 4\sin^2\left(\tfrac{\theta_1}{2}\right) - 4\sin^2\left(\tfrac{\theta_2}{2}\right)$$

$$A(k+1) = A(k)\left[1-\omega\left(\sin^2\left(\tfrac{\theta_1}{2}\right) + \sin^2\left(\tfrac{\theta_2}{2}\right)\right)\right]$$

$$G(\theta_1,\theta_2) = \left[1-\omega\left(\sin^2\left(\tfrac{\theta_1}{2}\right) + \sin^2\left(\tfrac{\theta_2}{2}\right)\right)\right]$$

2D Example: Jacobi

Analyze case $c \neq 0$

$$e_{jm}^{(k+1)} = \frac{\omega}{4+h^2 c_{jm}}\left(e_{jm-1}^{(k)} + e_{j-1m}^{(k)} + e_{j+1m}^{(k)} + e_{jm+1}^{(k)}\right) + (1-\omega)e_{jm}^{(k)}$$

$$e_{jm}^{(k+1)} = A(k)\exp\imath[j\theta_1 + m\theta_2]\left(1-\omega+\tfrac{\omega}{4+h^2c}(2\cos\theta_2 + 2\cos\theta_1)\right)$$

$$G(\theta_1,\theta_2) = \left(1-\omega+\tfrac{\omega}{4+h^2c}(2\cos\theta_2 + 2\cos\theta_1)\right)$$

Alternatively

$$G(\theta_1,\theta_2) = \left(1-\omega+\tfrac{\omega}{4+h^2c}(2\cos\theta_2 + 2\cos\theta_1)\right) =$$

$$1-\omega+\tfrac{\omega}{4}(2\cos\theta_2 + 2\cos\theta_1) - \tfrac{\omega}{4}(\cdots) + \tfrac{\omega}{4+h^2c}(\cdots) =$$

$$G_0(\theta_1,\theta_2) + (2\cos\theta_2 + 2\cos\theta_1)\left(-\tfrac{\omega}{4} + \tfrac{\omega}{4+h^2c}\right) =$$

$$G_0(\theta_1,\theta_2) - \tfrac{\omega}{4}\left(\tfrac{h^2c}{4+h^2c}\right)(2\cos\theta_2 + 2\cos\theta_1)$$

Convection-diffusion example

We solve several instances of the following equation

$-u_{xx}+bu_x+cu=f$

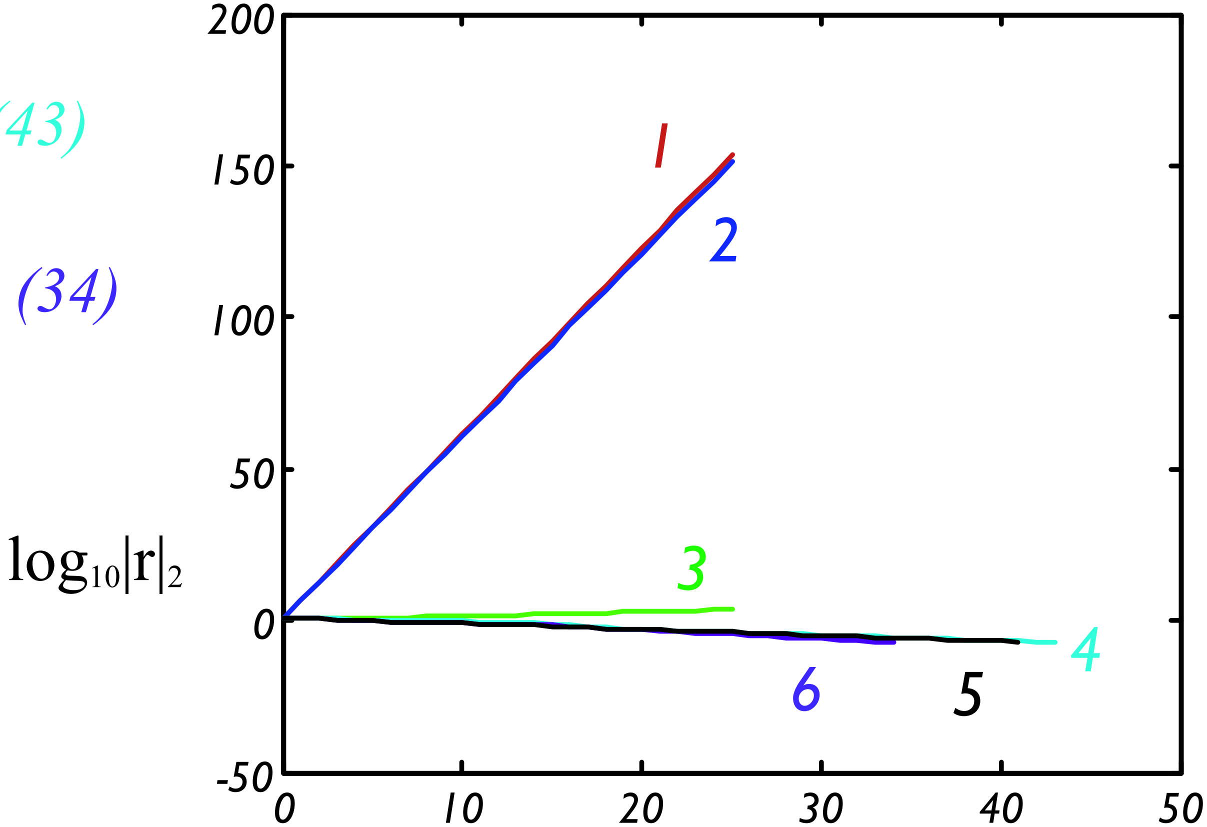
for various values of b, c, *and* h, *and various choices in the multigrid algorithm.*

We show how smoothing analysis helps guide choices

Convection-diffusion example

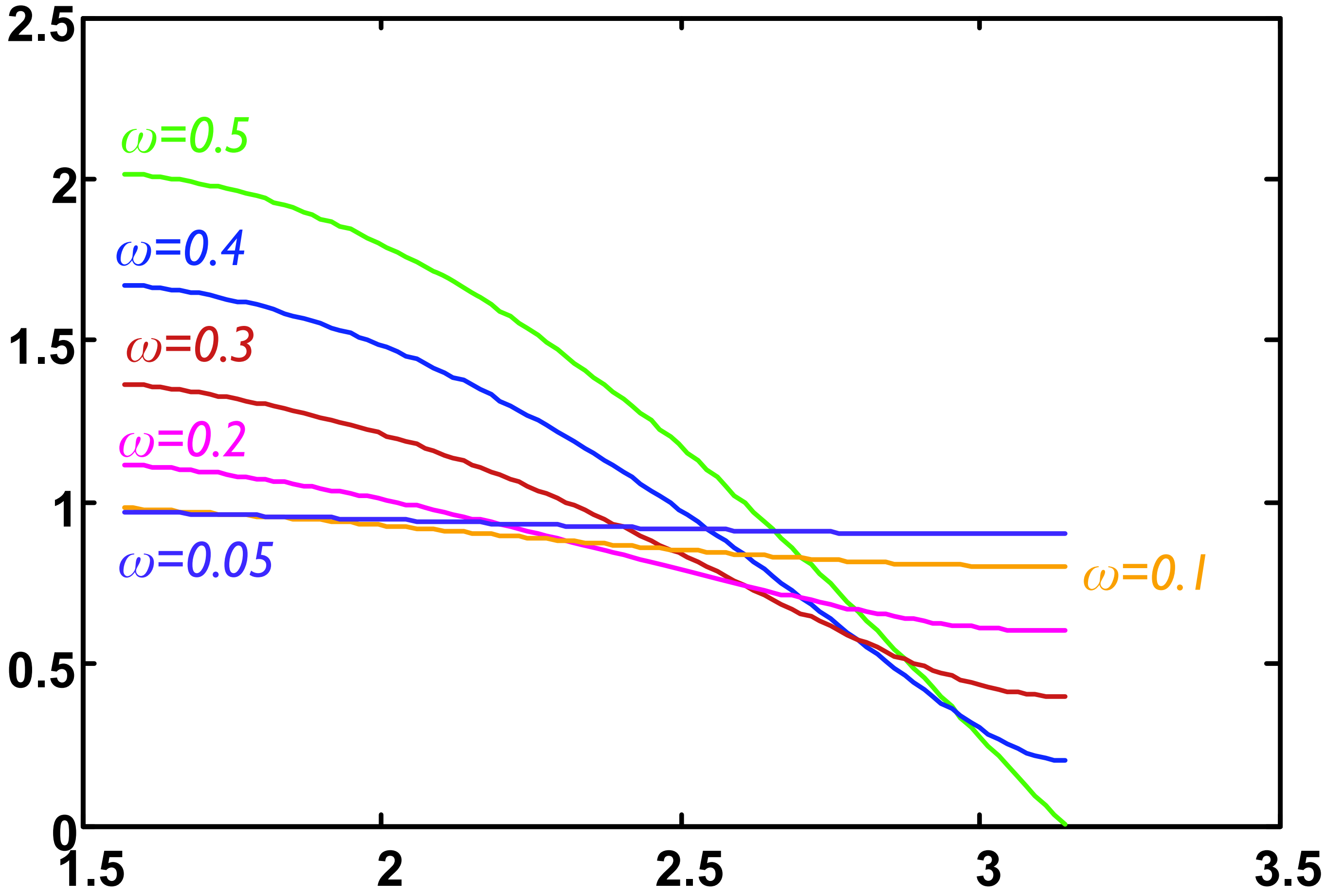
$-u_{xx}+500u_x=0$, $h=1/64$, nrel=10,

1. $\omega=0.5$, *injection*
2. $\omega=0.5$, *full weighting*
3. $\omega=0.1$, *injection*
4. $\omega=0.1$, *full weighting (43)*
5. $\omega=0.05$, *injection (41)*
6. $\omega=0.05$, *full weighting (34)*

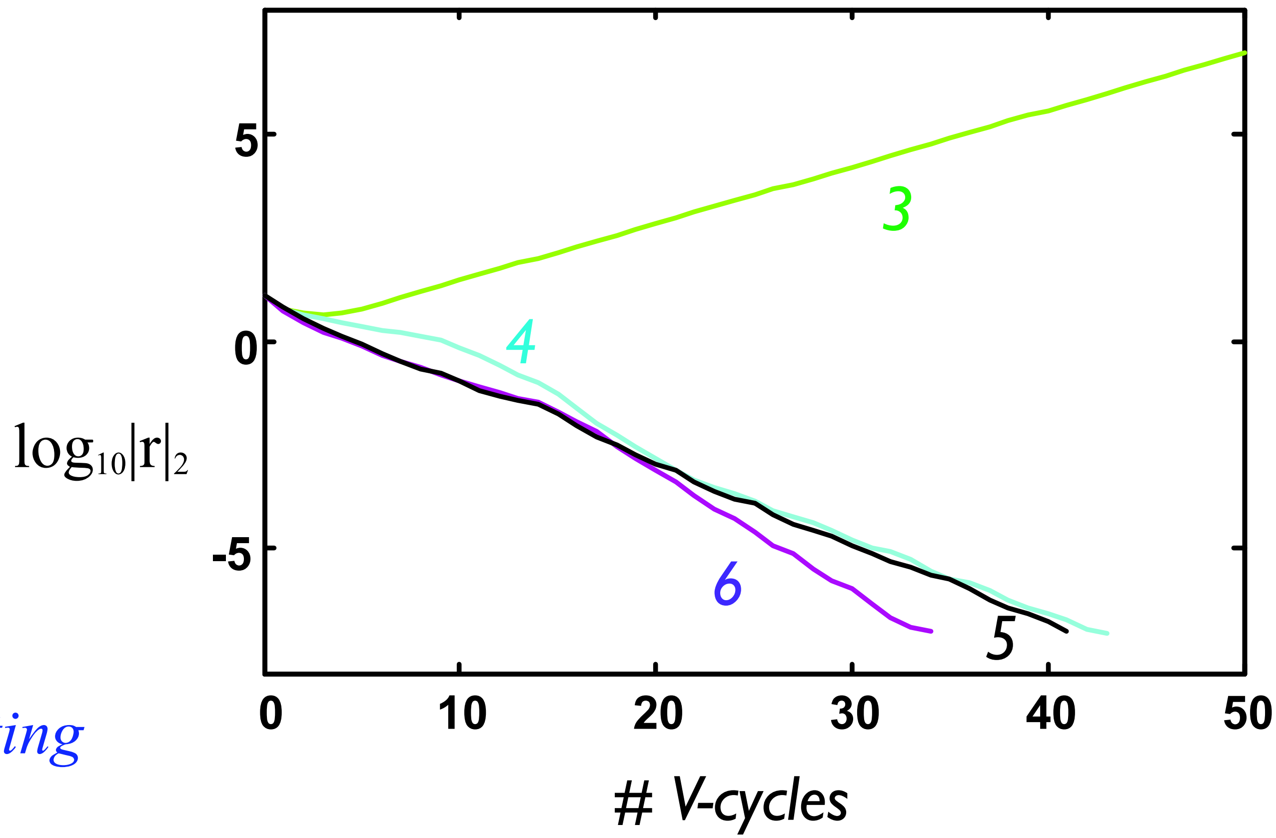


Convection-diffusion example

Amplification factors for the oscillatory modes and various weights



Convection-diffusion example



1. *w=0.5, injection*
2. *w=0.5, full weighting*
3. *w=0.1, injection*
4. *w=0.1, full weighting (43)*
5. *w=0.05, injection (41)*
6. *w=0.05, full weighting (34)*

Convection-diffusion example

Amplification factors for the oscillatory modes,
$\omega=0.1$*, injection, and various values for* c

c=10
c=1000
c=10000