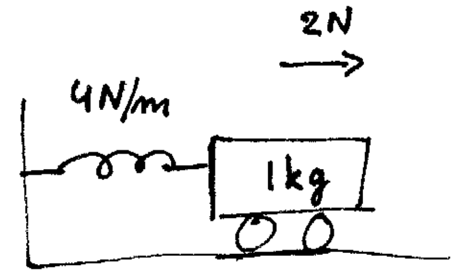
Example



position $x(t)$

velocity $\dot{x}(t)$

acceleration $\ddot{x}(t)$

friction $-k\dot{x}$

forces: $2 - 4x - k\dot{x}$

$\ddot{x} = F/m = 2 - 4x - k\dot{x}$ or $\ddot{x} + k\dot{x} + 4x = 2$

with $x(0) = \dot{x}(0) = 0$ (take $k = 5$)

Homogeneous problem: $\ddot{x} + 5\dot{x} + 4x = 0$

substitute $x = ce^{\lambda t} \rightarrow ce^{\lambda t}(\lambda^2 + 5\lambda + 4) = 0$

$(\lambda+4)(\lambda+1) = 0 \rightarrow \lambda_1 = -4,\ \lambda_2 = -1$

general solution homogeneous system: $x = c_1 e^{-4t} + c_2 e^{-t}$

particular solution of inhomogeneous system: $x = \frac{1}{2}$

$x = at^2 + bt + c \rightarrow \dot{x} = 2at + b \rightarrow \ddot{x} = 2a$

$\ddot{x} + 5\dot{x} + 4x \rightarrow 2a + 10at + 5b + 4at^2 + 4bt + 4c$

$= 4at^2 + (10a + 4b)t + 2a + 5b + 4c$

$$\rightarrow \begin{cases} 4a = 0 & \rightarrow a = 0 \\ 10a + 4b = 0 & \rightarrow b = 0 \\ 2a + 5b + 4c = 2 & \rightarrow 4c = 2 \rightarrow c = 1/2 \end{cases}$$

(guessing was faster here)

solution: $x(t) = c_1 e^{-4t} + c_2 e^{-t} + \frac{1}{2}$

fit boundary conditions

$$x(0) = c_1 + c_2 + 1/2 = 0$$
$$\dot{x}(0) = -4c_1 - c_2 = 0$$
$$\rightarrow \begin{cases} c_1 + c_2 = -1/2 \\ -4c_1 - c_2 = 0 \end{cases} \rightarrow \begin{cases} c_1 + c_2 = -1/2 \\ -3c_1 = -1/2 \end{cases}$$

$$c_1 = 1/6$$
$$c_2 = -2/3$$

$$x(t) = \tfrac{1}{6}e^{-4t} - \tfrac{2}{3}e^{-t} + \tfrac{1}{2} \qquad (x = \tfrac{1}{2} \text{ steady state})$$

Transform problem into system of first order equations:

$$\begin{cases} x_1 = x \\ x_2 = \dot{x}_1 = \dot{x} \end{cases} \rightarrow \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ 2 - 4x_1 - 5x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

2nd order ODE $\rightarrow$ system of first order ODEs.

Always possible! $\rightarrow$ Std software for systems of first order ODEs
(not always best approach)

$$\frac{d^n y}{dt^n} = f(t, \dot{y}, \ddot{y}, y^{(3)}, \dots, y^{(n-1)}) \rightarrow$$

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ \vdots \\ f(t, y^{(1)}, y^{(2)}, \dots, y^{(n-1)}) \end{pmatrix}$$

$\frac{d}{dt} x(t) = A\,x(t)$, A does not depend on t

Try simple solution $x(t) = g(t)\,v \rightarrow$

$$\dot{x} = \dot{g} v = A g v = g A v \iff$$

$$\dot{g}/g \; v = A v$$

Although lhs depends in principle on t, this cannot be the case as rhs does not depend on t. $\rightarrow$

$\dot{g}/g = \lambda$ constant

$\dot{g}/g = \lambda \rightarrow \ln g(t) = \lambda t + c \rightarrow g(t) = c_1 e^{\lambda t}$

Also $\dot{g}/g\, v = \lambda v = A v$ (eigenvalue problem)

$\begin{cases} \lambda \text{ eigenvalue} \\ v \text{ eigenvector} \end{cases} \rightarrow$ eigenpair

solution $x(t) = g(t)\,v = c_1 e^{\lambda t} v$

This can be done for any eigenpair: $A v_i = \lambda_i v_i$

$\rightarrow$ solution component $g_i(t)\,v_i = c_i e^{\lambda_i t} v_i$

$x(t) = \sum_i c_i e^{\lambda_i t} v_i$

$x(0) = x_0 = \sum_i c_i v_i \rightarrow$ decomposition of initial solution along eigenvectors.

$\frac{d}{dt} x(t) = A\,x(t) = \sum_i c_i \lambda_i e^{\lambda_i t} v_i$

$\frac{d}{dt} x(t) = f(x)$, where $f: \mathbb{R}^n \to \mathbb{R}^n$

(some autonomous dynamical system)

$\hat{x} : f(\hat{x}) = 0 \to \frac{d}{dt}\hat{x} = 0$ stationary point?

$\tilde{x} = \hat{x} + \varepsilon(t) \to$

$$\frac{d}{dt}\tilde{x}(t) = \frac{d}{dt}(\hat{x} + \varepsilon(t)) = \dot{\varepsilon}(t) = f(\hat{x}+\varepsilon) =$$

$$\underbrace{\cancel{f(\hat{x})}}_{=0} + J\varepsilon + O(\|\varepsilon\|^2)$$

$$J = \begin{pmatrix} \partial f_1/\partial x_1 & \cdots & \partial f_1/\partial x_n \\ \vdots & & \vdots \\ \partial f_n/\partial x_1 & \cdots & \partial f_n/\partial x_n \end{pmatrix} \to \text{Jacobian of } f \text{ at } \hat{x}$$

So, perturbation from $\hat{x}$ behaves (for small perturbations) as $\dot{\varepsilon}(t) = J\varepsilon(t)$.

Compute general solution as before. Eigenvalues of J play key role. $\varepsilon(t) = \sum_i c_i e^{\lambda_i t} v_i$ $\quad (J v_i = \lambda_i v_i)$

System is asymptotically stable if $\varepsilon \to 0$ for $t \to \infty$ (for small enough $\varepsilon(t)$)

Mode i, $c_i e^{\lambda_i t} v_i$, decays if $Re(\lambda_i) < 0$, constant magnitude if $Re(\lambda_i) = 0$ (neutrally stable).

If $Im(\lambda_i) \neq 0$ we get oscillations.

$$\lambda_j = \lambda_j^R + i\lambda_j^I \to e^{(\lambda_j^R + i\lambda_j^I)t} = e^{\lambda_j^R t}(\cos\lambda_j^I t + i \sin\lambda_j^I t)$$

Basics

$A \in \mathbb{C}^{n \times n}$

v eigenvector, λ eigenvalue if

$$Av = \lambda v \quad (v \neq 0)$$

(λ, v) eigenpair

$Av = \lambda v \iff (A - \lambda I)v = 0$ for some $v \neq 0$

So, $A - \lambda I$ <u>singular</u> $\to \det(A - \lambda I) = 0$

λ eigenvalue of A if and only if $\det(A - \lambda I)$ singular

$\to$ characteristic equation

$\to$ characteristic polynomial of A (of degree n)

Eigenvalues are roots of char. pol.

If $\hat{\lambda}$ is a multiple root of multiplicity k

$$\det(A - \lambda I) = (\lambda - \hat{\lambda})^k P_{n-k}(\lambda)$$

we say $\hat{\lambda}$ has <u>algebraic multiplicity</u> k

If $A \in \mathbb{C}^{n \times n}$ has n distinct eigenvalues, then A has n independent eigenvectors (proof later)

$\rightarrow$ semisimple

If A has fewer than n independent eigenvectors, A is called <u>defective</u>.

Let A have eigenvalue λ of <u>algebraic</u> <u>multiplicity</u> k. There can be at most k independent eigenvectors associated with λ. Fewer possible.

Number of independent eigenvectors associated with eigenvalue λ is called its <u>geometric</u> <u>multiplicity</u>.

$\rightarrow$ dim Null$(A - \lambda I)$

Generalization of eigenvector : invariant subspace.

S is invariant subspace of A if

$$\forall x \in S : \quad Ax \in S$$

Each eigenvalue has an associated invariant subspace whose dimension equals algebraic multiplicity.

Jordan decomposition of $A \in \mathbb{C}^{n \times n}$:

For every $A \in \mathbb{C}^{n \times n}$ there exists nonsingular $X \in \mathbb{C}^{n \times n}$ such that

$$X^{-1} A X = J = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{pmatrix}$$

~~where~~ where $J_i = \begin{pmatrix} \lambda_1 & 1 & & \\ & \lambda_i & 1 & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}^{m_i \times m_i}$

and $m_1 + m_2 + \cdots + m_k = n$.

$A = X J X^{-1}$

Each block has <u>one</u> corresponding eigenvector $\rightarrow$ k independent eigenvectors for A.

Each block has $m_i - 1$ principal vectors (of grade $2, 3, \ldots, m_i$)

If $m_i = 1$ (for all i), matrix is diagonalizable $\rightarrow$ full set of n eigenvectors.

$$A x_1 = X J X^{-1} x_1 = X J e_1 = X e_1 \lambda_1 = \lambda_1 x_1 \quad (m_1 = 5)$$

$$A x_2 = X J e_2 = X(e_2 \lambda_1 + e_1) = \lambda_1 x_2 + x_1$$

$$\vdots$$

$$A x_5 = \lambda_1 x_5 + x_4$$

$$A x_6 = \lambda_2 x_6$$

<u>etc</u>

Jordan decomposition (continued)

Multiple (Jordan) blocks can have the same eigenvalue: $\lambda_i = \lambda_j$

Sum of sizes of all blocks with same eigenvalue λ is algebraic multiplicity of λ.

The number of blocks with same eigenvalue λ is the geometric multiplicity of λ (because 1 eigenvector per block)

$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ has eig. value 2 with multiplicity 2 and 2 eig. vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ (for example).

Note, any vector $\alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ also eig. vector

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has double eigenvalue $\lambda = 1$

Solve for eigenvectors: $\begin{pmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\rightarrow \begin{pmatrix} y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow y = 0$

So, $\begin{pmatrix} x \\ 0 \end{pmatrix}$ eigenvector, but no other independent eigenvectors

algebraic multiplicity 2

geometric multiplicity 1 $\rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ <u>defective</u>

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \rightarrow \det(A-\lambda I) = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} =$$

$$\lambda^2 - 4\lambda + 4 - 1 = \cancel{\ldots} \; \lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda-3)(\lambda-1) = 0 \rightarrow \lambda = 3 \text{ or } \lambda = 1$$

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \rightarrow \det(A-\lambda I) = \begin{vmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 4 + 1$$

$$\lambda^2 - 4\lambda + 5 = 0 \rightarrow \lambda_{1,2} = 2 \pm \sqrt{16-20}$$

$$\rightarrow \lambda_{1,2} = 2 \pm 2i \quad (\text{complex!})$$

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \rightarrow \lambda(A) = \lambda(A_{11}) \cup \lambda(A_{22}) \quad \left(A = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} \right)$$

→ deagonal

→ upper triangular / lower triangular

Since there is not finite formula for roots of polynomial degree ≥ 5, there cannot be a "direct" algorithm for computing eigenvalues of matrix dim ≥ 5.

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \longrightarrow \lambda(A) = \lambda(A_{11}) \cup \lambda(A_{22})$$

can be shown using determinant.

Alternative:

a) if $A_{11} v_1 = \lambda_1 v_1 \longrightarrow \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ 0 \end{pmatrix} = \lambda_1 \begin{pmatrix} v_1 \\ 0 \end{pmatrix}$

b) assume $\lambda \in \lambda(A_{22}) \Rightarrow \lambda \notin \lambda(A_{11})$

let $A_{22} v_2 = \lambda v_2 \longrightarrow$

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} A_{11} v_1 + A_{12} v_2 \\ \lambda v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$A_{11} v_1 + A_{12} v_2 = \lambda v_1 \Leftrightarrow (A_{11} - \lambda I) v_1 = -A_{12} v_2$$

$$\Rightarrow v_1 = -(A_{11} - \lambda I)^{-1} A_{12} v_2$$

$\hookrightarrow$ possible since $\lambda \in \lambda(A_{11})$

vector $\begin{pmatrix} -(A_{11} - \lambda I)^{-1} A_{12} v_2 \\ v_2 \end{pmatrix}$ is eigenvector of A with eigenvalue λ

$$A = \begin{pmatrix} -a_1 & -a_0 \\ 1 & 0 \end{pmatrix} \rightarrow \det(A - \lambda I) = \begin{vmatrix} -a_1 - \lambda & -a_0 \\ 1 & -\lambda \end{vmatrix} =$$

$$\lambda^2 + a_1 \lambda + a_0$$

$$A = \begin{pmatrix} -a_3 & -a_2 & -a_1 & -a_0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \rightarrow \det(A - \lambda I) =$$

$$\begin{vmatrix} -a_3 - \lambda & -a_2 & -a_1 & -a_0 \\ 1 & -\lambda & 0 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & 1 & -\lambda \end{vmatrix} = (-a_3 - \lambda) \begin{vmatrix} -\lambda & 0 & 0 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix}$$

$$-1 \begin{pmatrix} -a_2 & -a_1 & -a_0 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{pmatrix}$$

$$= (-a_3 - \lambda)(-\lambda)^3 - \left[(-a_2) \begin{vmatrix} -\lambda & 0 \\ 1 & -\lambda \end{vmatrix} - \begin{vmatrix} -a_1 & -a_0 \\ 1 & -\lambda \end{vmatrix} \right.$$

$$= \lambda^4 + a_3 \lambda^3 - \left[-a_2 \lambda^2 - a_1 \lambda - a_0 \right]$$

$$= \lambda^4 + a_3 \lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0$$

~~roots of~~ eigenvalues of A are roots of

$$p(\lambda) = a_0 + a_1 \lambda + a_2 \lambda^2 + \cdots + a_{n-1} \lambda^{n-1} + \lambda^n$$

Can be done for any polynomial after scaling to be monic polynomial

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Let $A v_1 = d_1 v_1$ $\quad$ $d_1 \neq d_2$ $\quad$ $(v_1, v_2 \neq 0)$
$A v_2 = d_2 v_2$

assume $\underbrace{\alpha v_1 + \beta v_2}_{=0} = 0$ $\quad$ (α, β not ~~both~~ zero)

$A(\alpha v_1 + \beta v_2) = \alpha d_1 v_1 + \beta d_2 v_2 = 0$

However, $v_1 = -\beta/\alpha \, v_2$ $\quad$ (or $v_2 = -\alpha/\beta \, v_1$)

subst. $-\beta d_1 v_2 + \beta d_2 v_2 = \beta(d_2 - d_1) v_2 = 0$

but $\beta \neq 0$, $d_2 - d_1 \neq 0$, $v_2 \neq 0$ $\rightarrow$ contradict.

v_1, v_2 independent

let $d_3 \neq d_1$ and $d_3 \neq d_2$ by same argument
v_1, v_2 indep and v_1, v_3 indep.

Assume $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$ (where $\alpha_i \neq 0$, see above)

$A(\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3) = \alpha_1 d_1 v_1 + \alpha_2 d_2 v_2 + \alpha_3 d_3 v_3 = 0$

$$\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 0 \quad \text{and} \quad \begin{pmatrix} \quad \end{pmatrix} \begin{pmatrix} \alpha_1 d_1 \\ \alpha_2 d_2 \\ \alpha_3 d_3 \end{pmatrix} = 0$$

However $\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \alpha_1 d_1 \\ \alpha_2 d_2 \\ \alpha_3 d_3 \end{pmatrix}$ indep. $\rightarrow$ V rank 1 contradiction

Proof of $A \in \mathbb{C}^{m \times m}$ has n distinct eigen values then it has (at least) n independent eigenvectors.

→ We already showed that each selection of 2 eigenvectors is independent (set).

→ Assume this holds for each selection of $(n-1)$ eigenvectors and show it must hold for n as well.

If set of n vectors dependent, but not any set of $n-1$, then $\exists c \in \mathbb{C}^n : Vc = 0$ and $c_i \neq 0 \quad i = 1..n$

$V = [v_1 \, v_2 \cdots v_n]$ matrix with eig. vectors as columns

$$AV = [Av_1 \,..\, Av_n] = [v_1 d_1 \,..\, v_n d_n] = [v_1 .. v_n]\begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} = V\Lambda$$

$$Vc = 0 \rightarrow AVc = V\Lambda c = V\begin{pmatrix} d_1 c_1 \\ \vdots \\ d_n c_n \end{pmatrix} = 0$$

$$A^2 Vc = V\Lambda^2 c = V\begin{pmatrix} d_1^2 c_1 \\ \vdots \\ d_n^2 c_n \end{pmatrix} = 0 \quad \text{etc } (A^{n-1}Vc)$$

$$\Rightarrow V\begin{bmatrix} c_1 & d_1 c_1 & d_1^2 c_1 & & c_1 d_1^{n-1} \\ c_2 & d_2 c_2 & d_2^2 c_2 & \cdots & c_2 d_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ c_n & d_n c_n & d_n^2 c_n & & c_n d_n^{n-1} \end{bmatrix} = 0$$

$$\text{but } \begin{bmatrix} c_1 & c_1 d_1 & d_1^2 c_1 & \cdots & d_1^{n-1} c_1 \\ c_2 & c_2 d_2 & & \cdots & d_2^{n-1} c_2 \\ & & \vdots & & \end{bmatrix}$$ <u>nonsingular</u> →

V must be zero matrix → contradiction.

So, we must have $c_i = 0$ for at least one coefficient, but then there is a selection of $n-1$ vectors dependent. (contradict.)

→ $\{v_1 .. v_n\}$ independent

$A \in \mathbb{C}^{n\times n}$, n independent eigenvectors

$$\begin{cases} A v_1 = v_1 \lambda_1 \\ A v_2 = v_2 \lambda_2 \\ \vdots \end{cases} \rightarrow A[v_1\ v_2 \cdots v_n] = [v_1\ v_2 \cdots v_n]\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

in matrix notation $AV = V\Lambda$ (eigendecomposition of A)

since all v_i independent V nonsingular

$A = V\Lambda V^{-1}$ and $V^{-1}AV = \Lambda$ $\rightarrow$ <u>A diagonalizable</u>

$$A v_i = V\Lambda \underbrace{V^{-1} v_i}_{e_i} = [v_1\ v_2 \cdots v_n]\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} e_i = v_i \lambda_i$$

since $V e_i = v_i$

(it works!)

Consider again $\frac{d}{dt} x(t) = A x(t) = V\Lambda V^{-1} x(t)$

$\eta(t) = V^{-1} x(t)$ (and hence $x(t) = V\eta(t)$ $\longrightarrow$

$V \frac{d}{dt}\eta(t) = V\Lambda\eta(t) \overset{V \text{ constant}}{\Longrightarrow} \frac{d}{dt}\eta(t) = \Lambda\eta(t)$

$\rightarrow$ decoupled system of ODEs : $\dot{\eta}_i(t) = \lambda_i \eta_i(t)$

solution $\eta_i(t) = \eta_i(t_0) \cdot e^{\lambda_i(t-t_0)}$ (for $i = 1..n$)

$\hookrightarrow$ $x(t) = V\eta(t) = \sum_{i=1}^{n} v_i\, \eta_i(t_0) \cdot e^{\lambda_i(t-t_0)}$

$x_0 = x(t_0)$ given $\rightarrow$ $x_0 = V\eta(t_0) \rightarrow \eta(t_0) = V^{-1} x_0$

(so initial condition defines $\eta(t_0)$)