Eigenvalue problems - Basics /

$A \in \mathbb{C}^{n \times n}$ (or $\mathbb{R}^{n \times n}$)

(λ, x) is (right) eigenpair if $Ax = \lambda x$

(λ, y) is left eigenpair if $y^H A = \lambda y^H$

The spectrum of A is the (multi) set of all eigenvalues of A (including multiplicities)

Note that eig. vector only unique up to multiplication by scalar (for a simple eigenvalue).

→ we will often assume some normalization

e.g. $\|x\| = 1$, $y^H x = 1$, ... (as useful)

For some matrices finding eigenpairs easy:

1) $A = \mathrm{diag}(a_{ii}) = \begin{pmatrix} a_{11} & & \\ & a_{22} & \\ & & \ddots \end{pmatrix}$

$A e_i = a_{ii} e_i \quad i = 1..n$

2) A triangular (upper or lower)

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & a_{23} \cdots & \vdots \\ & & \ddots & \vdots \\ & & & a_{nn} \end{pmatrix} \qquad A e_1 = a_{11} e_1$$

$$Ax = \lambda x \iff (A - \lambda I)x = 0 \to \det(A - \lambda I) = 0$$

$$\to (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda) = 0 \to \lambda = a_{ii} \text{ roots}$$

So, diagonal coefficients are eigenvalues

Eigenvectors ?

$$\begin{pmatrix} A_{11} & a_{1k} & A_{1,k+1} \\ & a_{kk} & a_{k,k+1} \\ & & A_{k+1,k+1} \end{pmatrix} \begin{pmatrix} x_1 \\ \xi_k \\ 0 \end{pmatrix} = \overset{a_{kk}}{\cancel{\lambda}} \begin{pmatrix} x_1 \\ \xi_k \\ 0 \end{pmatrix}$$

clear that eig. value $\lambda = a_{kk}$.

$$\begin{cases} A_{11} x_1 + a_{1k} \xi_k = x_1 a_{kk} \\ a_{kk} \xi_k = a_{kk} \xi_k \quad \rightarrow \text{always true} \end{cases}$$

$$(A_{11} - a_{kk} I) x_1 = - a_{1k} \xi_k$$

if a_{kk} not an eigenvalue of A_{11} then always solvable $\rightarrow$ $x_1 = -\xi_k (A_{11} - a_{kk} I)^{-1} a_{1k}$

So, ξ_k arbitrary (except $\xi_k \neq 0$)

If a_{kk} eigenvalue of A_{11} then

still solvable if $a_{1k} \in \text{range}(A_{11} - a_{kk} I)$
$\rightarrow$ eig. vector exist & can be computed

not solvable if $a_{1k} \notin \text{range}(A - a_{kk} I)$
$\hookrightarrow$ eigenvector does not exist
(discuss this later)

$$\begin{pmatrix} A_{11} & a_{1k} & a_{1k+1} & A_{1k+2} \\ & a_{kk} & a_{kk+1} & a_{kk+2} \\ & & a_{k+1k+1} & a_{k+1k+2} \\ & & & A_{k+2,k+2} \end{pmatrix} \quad \begin{array}{l} \text{where } a_{kk} = a_{k+1k+1} \\ a_{kk} \notin \Lambda(A_{11}) \end{array}$$

$$\rightarrow \left(a_{kk}, \begin{pmatrix} -(A_{11} - a_{kk} I)^{-1} a_{1k} \\ 1 \\ 0 \end{pmatrix} \right) \text{ eig. pair}$$

$$a_{k+1k+1} \rightarrow \begin{pmatrix} A_{11} - a_{kk} I & a_{1k} \\ & 0 \end{pmatrix} x = - \begin{pmatrix} a_{1k+1} \\ a_{kk+1} \\ \cancel{0} \end{pmatrix}$$

This equation has solution __iff__ $a_{kk+1} = 0$

$(a_{kk} = a_{k+1k+1})$

(1) $A_{11} x_1 + a_{1k} \xi_1 + a_{1k+1} \xi_2 = a_{kk} x_1$

(2) $\cancel{a_{kk} \xi_1} + a_{kk+1} \xi_2 = \cancel{a_{kk} \xi_1} \; 0$

(3) $a_{kk} \xi_2 = a_{kk} \xi_2$ (true)

(2) $\Rightarrow a_{kk+1} = 0$ __or__ $\xi_2 = 0$

if $\xi_2 = 0 \rightarrow$ get same eig. vec. as for a_{kk}

if $\xi_2 \neq 0$ __need__ $a_{kk+1} = 0$

(intuitively) simple case: A singular $\rightarrow (0, x)$

s.t. $Ax = 0$ $(x \neq 0)$. Such x exists by definition

$Ax = dx \iff (A - dI)x = 0 \rightarrow (d, x)$

$A - dI$ singular $\iff \det(A - dI) = 0 \rightarrow$

polynomial of degree n: characteristic poly.

eig. values of A are roots, $d_1 \ldots d_n$ of $\det(A - dI) = 0$

poly of degree n always has n roots, may be repeated

$P_A(d) = 0 \rightarrow P_A(d) = (d - d_1)^{m_1} (d - d_2)^{m_2} \cdots (d - d_k)^{m_k}$

where k is number of distinct eig. values and

m_i are __algebraic__ multiplicities

$m_1 + m_2 + \cdots + m_k = n$

Note that $y^H(A-\lambda I)=0 \iff y^H A = \lambda y^H$

So, "left" and "right" eig. values are the same :

every eigenvalue has left and right eigen value.

note each (multiple) eig. value has <u>at least one</u> right and left eig. vector.

A block (upper or lower) triangular

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ & A_{22} & \cdots & A_{2k} \\ & & \ddots & \vdots \\ & & & A_{kk} \end{pmatrix}$$

$$\det(A-\lambda I) = \det(A_{11}-\lambda I)\det(A_{22}-\lambda I)\cdots = 0 \iff$$

$$\det(A_{11}-\lambda I)=0 \ \underline{\text{or}}\ \det(A_{22}-\lambda I)=0 \ \underline{\text{or}} \cdots$$

also $p_A(\lambda) = p_{A_{11}}(\lambda)\, p_{A_{22}}(\lambda) \cdots p_{A_{kk}}(\lambda)$

So, $\Lambda(A) = \Lambda(A_{11}) \cup \Lambda(A_{22}) \cup \cdots \cup \Lambda(A_{kk})$

(counting multiplicities)

$A \in \mathbb{R}^{n\times n} \to p_A(\lambda)$ has real coefficients $\Rightarrow$

eigenvalues are real or come in complex conjugate pairs

$\lambda \in \Lambda(A) \Rightarrow \lambda \in \mathbb{R}$ <u>or</u> $\bar{\lambda} \in \Lambda(A)$

~~Let $A-\lambda I$ be singular and let $\{x_1, \ldots, x_k\}$ be~~

~~the set of all eigenvectors associated with λ:~~

~~$Ax_i = \lambda x_i \quad i=1\ldots k$~~

Let $A - \lambda I$ be singular

- if $x \in \text{null}(A - \lambda I) \rightarrow (A - \lambda I)x = 0$

  then x eigenvector of A with eigenvalue λ

- if $k = \dim \text{null}(A - \lambda I)$ then there exists

  <u>basis</u> $\{x_1, \dots, x_k\}$ for $\text{null}(A - \lambda I)$

  hence $x \in \text{null}(A - \lambda I) \Rightarrow x = \sum_i \alpha_i x_i$ (some set $\alpha_1 \dots \alpha_k$)

  (there are infinitely many eig. vec.s ~~associ~~ with eig. value λ but we can pick only k indep. ones)

- k is called geometric multiplicity of λ
- if m is algebraic multiplicity of λ then

  $k \leq m$

  * if $k = m$ λ has <u>complete</u> set of eigenvectors associated with it.
  * if $k < m$ we call λ and A <u>defective</u>
  * it follows from Schour decomp (later) that $k > m$ impossible.

Defective matrices:

$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \left(0, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$ eig. pair

but there exists no other eig. vector x for eig. val. 0

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x_2 = 0 \\ 0 \cdot x_1 = 0 \end{cases} \rightarrow \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \text{ dependent with } \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Similarity transformation:

$U \in \mathbb{C}^{n\times n}$ nonsingular: $B = U^{-1}AU$ similarity transformation

B similar to A $\rightarrow$ $\Lambda(B) = \Lambda(A)$ (incl. multipl.)

$Ax = \lambda x \quad (\lambda \in \Lambda(A))$

$BU^{-1}x = U^{-1}AU \cdot U^{-1}x = U^{-1}(\lambda x) = \lambda U^{-1}x$

So, $\lambda \in \Lambda(B)$ and $U^{-1}x$ is corresponding eig. vector.

If x_1 and x_2 are indep eig. vec.s assocated with λ, ~~same~~ then $U^{-1}x_1$ and $U^{-1}x_2$ are eig. vec.s of B

Moreover, ~~[illegible]~~

$\det(\lambda I - B) = \det(\lambda U^{-1}U - U^{-1}AU)$

$= \det(U^{-1}(\lambda I - A)U) = \det(U^{-1})\det(\lambda I - A)\det(U)$

$= \det(\lambda I - A)$

So, B and A have same characteristic poly. $\rightarrow$ same eigenvalues with same multiplicities

If A has full/complete set of independent eig. vec.s

$Ax_i = \lambda_i x_i \quad i = 1..n$

$X = [x_1\ x_2 \cdots x_n]$ (col.s are eig. vec.s)

$AX = [Ax_1\ Ax_2 \cdots Ax_n] = [x_1\lambda_1\ x_2\lambda_2 \cdots x_n\lambda_n]$

$= X \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

$AX = X\Lambda \iff X^{-1}AX = \Lambda \iff A = X\Lambda X^{-1}$

(x_i are independent so X^{-1} exists)

So, A diagonalized by similarity transformation defined by its eig. vectors (A diagonalizable)

~ Unfortunately A (general) may not have full set of eig. vec.s

Moreover, X may be ill-conditioned (some eig. vec.s nearly dependent) $\rightarrow$

working with X^{-1} in practice problematic.

~ better form/decomposition that reveals eig. values and eig. vec.s with "modest" work.

Schur decomp. (theo 1.12). $A \in \mathbb{C}^{n\times n}$. There is unitary $U \in \mathbb{C}^{n\times n}$ s.t. $U^H A U = T$ where T is uppertriangular (by choice of U eig. vals of A may be in any order on diagonal of T)

Proof (by induction): let $\lambda_1, \lambda_2, \dots, \lambda_n$ be ordering eig. val.s of A, and $Ax_1 = \lambda_1 x_1$. (Assume theo true for all mat.s order $n-1$ and smaller)

unitary $X = [x_1 \;\; X_2]$

$$X^H A X = \begin{pmatrix} x_1^H A x_1 & x_1^H A X_2 \\ X_2^H A x_1 & X_2^H A X_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & x_1^H A X_2 \\ 0 & X_2^H A X_2 \end{pmatrix}$$

~~Using~~ By assumption Schur decomp. exists for $X_2^H A X_2$

$A = A^H$ (Hermitian, symm if real)

$T = U^H A U$ be Schur form (decomp) of A

then $T^H = U^H A^H U = U^H A U = T$

So, T uppertri. and lowertri $\rightarrow$ diagonal

Moreover, since $T^H = T \rightarrow$

$\mathrm{diag}(d_1, \dots, d_n) = \mathrm{diag}(\bar{d}_1, \dots, \bar{d}_n) \rightarrow$

all $d_i \in \mathbb{R}$

Also $AU = UT = U \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}$

So, columns of U are eigenvectors. (always complete set)

$A = U \Lambda U$ spectral decomp. (eigen decomp.)

$AA^H = A^H A$: A is normal matrix

(obviously an Hermitian matrix is normal)

others $A^H = -A$ (skew Hermitian)

$A^H A = A A^H = I$ (unitary)

theo A normal matrix has a complete set of orthonormal eigenvectors

Proof:

1) A normal and $\begin{pmatrix} L & H \\ G & M \end{pmatrix} = A$

then $\|G\|_F = \|H\|_F$

$A^H A = A A^H \Rightarrow$

$L^H L + G^H G = L L^H + H H^H$

~~avg~~ $\|B\|_F^2 = \sum_{i,j} |b_{i,j}|^2$

$\text{trace}(B^H B) = b_1^H b_1 + b_2^H b_2 + \dots + b_n^H b_n =$

$\sum_i |b_{i1}|^2 + \sum_i |b_{i2}|^2 + \dots = \|B\|_F^2$

similarly $\text{trace}(BB^H) = \|B\|_F^2$

So, taking trace left and right $\rightarrow$

$\|L\|_F^2 + \|G\|_F^2 = \|L\|_F^2 + \|H\|_F^2 \Rightarrow$

$\|G\|_F^2 = \|H\|_F^2$

2) $T = U^H A U \quad T^H T = U^H A^H \cancel{U U^H} A U =$

$U^H A^H A U = U^H A A^H U = T T^H$

T also normal - but also upper tri

$T = \begin{pmatrix} t_{11} & t_{12}^H \\ 0 & T_{22} \end{pmatrix} \overset{\text{then}}{\Rightarrow} t_{12}^H = 0$

T_{22} is normal too $\rightarrow$ T diagonal

(The coeff. $t_{11}, t_{22}, \dots$ are in general not real)

How close can we get to diagonal?

$$A = \begin{pmatrix} L & H \\ 0 & M \end{pmatrix} \rightarrow \text{sim. transform that gives block diagonal}$$

product of block uppertri mat.s is uppertri. again

$U^{-1} A U$ with $\begin{pmatrix} I & Q \\ 0 & I \end{pmatrix} \equiv U$

$$\underbrace{\begin{pmatrix} I & -Q \\ 0 & I \end{pmatrix} \begin{pmatrix} L & H \\ 0 & M \end{pmatrix}} \begin{pmatrix} I & Q \\ 0 & I \end{pmatrix} = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$$

$$\begin{pmatrix} L & H - QM \\ 0 & M \end{pmatrix} \begin{pmatrix} I & Q \\ 0 & I \end{pmatrix} = \begin{pmatrix} L & LQ - QM + H \\ 0 & M \end{pmatrix}$$

block diagonal if

$$LQ - QM = -H$$

Sylvester's eq.

$AX - XB = C$ (solve for X)

$A \in \mathbb{C}^{n \times n}$, $B \in \mathbb{C}^{m \times m}$

Sylvester operator $S : \mathbb{C}^{n \times m} \rightarrow \mathbb{C}^{n \times m}$

$SX := AX - XB \rightarrow SX = C$

<u>S is linear</u>

$$S(\alpha X_1 + \beta X_2) = A(\alpha X_1 + \beta X_2) - \cancel{B}(\alpha X_1 + \beta X_2)B$$

$$= \alpha A X_1 - \alpha X_1 B + \beta A X_2 - \beta X_2 B$$

$$= \alpha S X_1 + \beta S X_2$$

So, solution exists (unique) if S nonsingular

<u>theo 1.16</u> : S nonsingular iff (if and only if)

$$\Lambda(A) \cap \Lambda(B) = \emptyset$$

I: $Ax = \lambda x$, $y^H B = \mu y^H$ and let $X = xy^H$

Then $SX = Axy^H - xy^H B = \lambda X - \mu X = (\lambda - \mu) X$

If $\hat{\lambda} \in \Lambda(A) \cap \Lambda(B)$ ~~an~~, then $\hat{x}, \hat{y}$ exist s.t.

$A\hat{x} = \hat{\lambda}\hat{x}$, $\hat{y}^H B = \hat{\lambda}\hat{y}^H$ and hence

$\hat{X} = \hat{x}\hat{y}^H \neq 0$ and $S\hat{X} = 0$ (singular)

II (if $\Lambda(A) \cap \Lambda(B) = \emptyset \Rightarrow S$ nonsingular)

Solve $SX = C$ (show solution exists)

Let $T = V^H B V \rightarrow$ (T upper tri)

$$AXV - XBV = CV \Leftrightarrow A(XV) - (XV)V^H BV = (CV)$$

$\Leftrightarrow AY - YT = D$ with $Y = XV$, $D = CV$

columnwise: $Ay_1 - y_1 t_{11} = d_1 \Leftrightarrow$

$(A - t_{11} I) y_1 = d_1$ has solution because $t_1 \in \Lambda(B)$ and $\Lambda(A) \cap \Lambda(B) = \emptyset$

$\rightarrow$ compute y_1

$Ay_2 - y_1 t_{12} - y_2 t_{22} = d_2 \iff$

$(A - t_{22} I)\, y_2 = d_2 + y_1 t_{12}$

again, since $t_{22} \in \Lambda(B)$ and $\Lambda(A) \cap \Lambda(B) = \phi$

we know $t_{22} \notin \Lambda(A)$ and system solvable.

etc

Hence we can compute (solve for Y) for any D. $Y = XV \rightarrow X = YV^H$

So, we can compute X (X exists) for any C.

Therefore S nonsingular.

let $SX = \sigma X$.

$AX - XB = \sigma X \iff (A - \sigma I)X - XB = 0$

So that $S_\sigma X := (A - \sigma I)X - XB$ singular

$\Rightarrow \Lambda(A - \sigma I) \cap \Lambda(B) \neq \phi$

$\lambda \in \Lambda(A) \rightarrow \lambda - \sigma \in \Lambda(A - \sigma I)$

So, there must be $\lambda \in \Lambda(A)$, $\mu \in \Lambda(B)$ s.t.

$\lambda - \sigma = \mu \iff \sigma = \lambda - \mu$ (reverse holds as well)

Eigenvalues of S are diff.s of eig. vals of A and eig. val.s of B

~~Ex.~~ $A = \begin{pmatrix} L & H \\ 0 & M \end{pmatrix}$ and $\Lambda(L) \cap \Lambda(M) = \phi$

then Q exists s.t. $LQ - QM = -H$

and sim. transform

$$\begin{pmatrix} I & -Q \\ 0 & I \end{pmatrix} \begin{pmatrix} L & H \\ 0 & M \end{pmatrix} \begin{pmatrix} I & Q \\ 0 & I \end{pmatrix} = \begin{pmatrix} L & 0 \\ 0 & M \end{pmatrix}$$

Apply recursively.

let $\Lambda(A)$ be partitioned in disjoint multisets

$\Lambda(A) = h_1 \cup h_2 \cup \cdots \cup h_k$

then Schur decomp. exists s.t.

$T = X^H A X$ with eigenvalues in order of $h_1, h_2, \ldots$

$T = \begin{pmatrix} L_1 & H \\ 0 & M \end{pmatrix}$ and exist sim. transform $\hat{Q}$

$\hat{Q}^{-1} T \hat{Q} = \begin{pmatrix} L_1 & 0 \\ 0 & M \end{pmatrix}$

now apply same steps to get L_2

(seperated out of M) and so on.

theo 1.19 : let $\Lambda(A)$ be $h_1 \cup h_2 \cup \cdots \cup h_k$

where $h_i \cap h_j = \phi$ $i \neq j$. Then X exists s.t.

$X^{-1} A X = \mathrm{diag}(L_1, L_2, \ldots, L_k)$

and $\Lambda(L_i) = h_i$

Note we can apply this to partion h_i such that

each h_i has one eigen value only, repeated

with alg. multiplicity

Jordan Form / decomposition

Jordan block $J_m(\lambda) = \underbrace{\begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & \lambda \end{pmatrix}}_{m} \Big\} m \in \mathbb{C}^{m\times m}$

Note $J_m(\lambda)$ has exactly 1 eigenvector (e_1)

λ : alg. multiplicity m

geom " 1

Theo 1.22 : Let A have k distinct eigenvalues $\lambda_1 \dots \lambda_k$ with alg. mult. $m_1, m_2, \dots, m_k$

Then there are unique integers

$m_{ij} \quad i = 1 \dots k,\ j = 1 \dots l_i \ : \ m_i = \sum_{j=1}^{l_i} m_{ij}$

and nonsingular X s.t. $X^{-1}AX = \mathrm{diag}(J_1, \dots, J_k)$

where

$$J_i = \mathrm{diag}\left(J_{m_{i1}}(\lambda_i), J_{m_{i2}}(\lambda_i), \dots, J_{m_{il_i}}(\lambda_i) \right)$$

Take $X = (X_1 \dots X_k) \Rightarrow AX_i = X_i J_i$

$X_i = (X_{i1} \cdots X_{il_i}) \rightarrow A X_{ij} = X_{ij} J_{m_{ij}}(\lambda_i)$

$X_{ij} = \cancel{(x_{ij}^{(1)} \dots x_{ij}^{(m_{ij})})} \left(x_1^{(ij)}\ x_2^{(ij)} \dots x_{m_{ij}}^{(ij)} \right)$

$A x_1^{(ij)} = \lambda_i x_1^{(ij)}$

$A x_p^{(ij)} = \lambda_i x_p^{(ij)} + x_{p-1}^{(ij)} \qquad p = 2 \dots m_{ij}$

(Section 2, read def.s of norms)

<u>Operator norm</u> (induced matrix norm)

Let μ be vector norm on $\mathbb{C}^m$

ν " " " " $\mathbb{C}^n$

$$\|A\|_{\mu,\nu} := \max_{\nu(x)=1} \mu(Ax)$$

is matrix norm subordinate to μ and ν

---

rounding error: $fl(A)$ result of rounding elements of A

(assume IEEE arithm.)

$$fl(a_{ij}) = a_{ij}(1+\varepsilon_{ij}) \quad |\varepsilon_{ij}| \le \varepsilon_M$$

$$fl(A) = A + E \quad E = (a_{ij}\,\varepsilon_{ij})$$

$|E| \le |A|\,\varepsilon_M$ $\quad |\cdot|$ comp. wise abs value

absolute norm: $\||A|\| = \|A\| \to \|E\| \le \|A\|\,\varepsilon_M$

$\frac{\|E\|}{\|A\|} \le \varepsilon_M$ (normwise <u>relative</u> error)

Ex. $x = \begin{pmatrix} 1 \\ 10^{-3} \\ 10^{-5} \end{pmatrix}$ $\quad \tilde{x} = \begin{pmatrix} 1.00001 \\ 1.01 \cdot 10^{-3} \\ 2 \cdot 10^{-5} \end{pmatrix}$

$\frac{\|\tilde{x} - x\|}{\|x\|} = 10^{-5}$ but $\frac{|\tilde{x}_3 - x_3|}{|x_3|} = 1$ (100% error)

Def 2.3

Let $\|\cdot\|_{l,m}$, $\|\cdot\|_{m,n}$, $\|\cdot\|_{l,n}$ be matrix norms defined on $\mathbb{C}^{l\times m}$, $\mathbb{C}^{m\times n}$, and $\mathbb{C}^{l\times n}$. These norms are consistent if for all $A \in \mathbb{C}^{l\times m}$, $B \in \mathbb{C}^{m\times n}$, we have $\|AB\|_{l,n} \le \|A\|_{l,m} \cdot \|B\|_{m,n}$

$\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$, $\|\cdot\|_F$ are consistent norms

Many ways to construct useful norms
On a finite dim. vector space all norms are equivalent.

$A \in \mathbb{C}^{n\times n}$. $\rho(A) := \max\limits_{\lambda \in \Lambda(A)} |\lambda|$ is called spectral radius

$\rho(A)$ is not a norm, but closely related.

consider $\rho\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)$

(λ, x) with $|\lambda| = \rho$ is dominant eigenpair (vector, value)

Theo 2.7. $\rho(A) \le \|A\|$ for any consistent norm $\|\cdot\|$

Let (λ, x) be dom. eigenpair. Then

$\|Ax\| = \|\lambda x\| = |\lambda|\,\|x\|$ and
$\|Ax\| \le \|A\| \cdot \|x\|$

So, ~~$\|A\|$~~ $|\lambda| \le \|A\|$

Theo 2.8. $A \in \mathbb{C}^{n\times n}$. For any $\varepsilon > 0$ there is a consistent norm $\|\cdot\|_{A,\varepsilon}$ s.t.

$$\|A\|_{A,\varepsilon} \le \rho(A) + \varepsilon$$

If dom. eigenvalues nondefective, there is a consistent norm $\|\cdot\|_A$ s.t.

$$\|A\| = \rho(A)$$

Proof:

Schur form $U^H A U = \Lambda + \underbrace{N}_{\text{uppertri part}}$

$\eta > 0$ let $D_\eta = \mathrm{diag}(1, \eta, \eta^2, \cdots, \eta^{n-1})$

Consider $D_\eta^{-1} U^H A U D_\eta = \Lambda + D_\eta^{-1} N D_\eta$

$$(D_\eta^{-1} N D_\eta)_{ij} = \begin{cases} 0 & i \geq j \\ n_{ij}\, \eta^{j-i} \end{cases}$$

So, for $\eta < 1$ we can make upper tri part arbitrarily small (but not zero)

Pick η s.t. $\|D_\eta^{-1} N D_\eta\|_* \le \varepsilon$

$$\|B\|_{A,\varepsilon} = \|D_\eta^{-1} U^H B U D_\eta\|_\infty \le \|\Lambda\|_\infty + \|D_\eta^{-1} N D_\eta\|_\infty$$

$$\le \rho(A) + \varepsilon$$

Chapter 4

Generalize eigenvectors $\rightarrow$ Eigenspaces

$A \in \mathbb{C}^{n \times n}$, $\mathcal{X}$ subspace of $\mathbb{C}$ (subset and vector space)

Then $\mathcal{X}$ is invariant subspace or eigenspace if

$A\mathcal{X} := \{Ax : x \in \mathcal{X}\} \subset \mathcal{X}$

The span of an eigenvector is invariant subspace

let $A \in \mathbb{R}^{n \times n}$ with complex eig. pair

$(\lambda, x) \neq: \begin{cases} \lambda \in \mathbb{C} \ (\lambda \notin \mathbb{R}) \\ x = y + iz, \ y, z \in \mathbb{R} \end{cases}$

Then $A[y\ z] = [y\ z]\,L$ (all real)

where $\lambda, \bar{\lambda} \in \Lambda(L)$

So, we can work strictly with real vectors and matrices (up to 2×2 blocks)

Theo 1.2 p.243
let $\mathcal{X}$ be eigenspace and X (col.s) form basis for $\mathcal{X}$. Then

$$AX = XL, \quad L = X^I A X$$

where X^I is left inverse of X : $X^I X = I$

(but in general $XX^I \neq I$ or not even def.d)

L is representation of A on eigenspace $\mathcal{X}$ with basis X

let $x \in \mathcal{X}$ and $Ax = \lambda x$. Then

$x = Xu$ (X basis) and $Ax = AXu = XLu$

$Ax = \lambda x = \lambda Xu = XLu$

$X^I X(\lambda u) = X^I X L u \iff Lu = \lambda u$

(also works the other way around)

So, if (λ, x) eig. pair of A then $(\lambda, X^I x)$ eig. pair of L

and if (λ, u) eig. pair of L then (λ, Xu) eig. pair of A

There is a one-one correspondence between eigenvectors of A lying in $\mathcal{X}$ and eigenvectors of L

Exploit this when interested in few (relatively) eigenpairs of A

Def 1.3

$A \in \mathbb{C}^{n \times n}$, $X \in \mathbb{C}^{n \times k}$, $L \in \mathbb{C}^{k \times k}$

If X full ~~rand~~ rank and $AX = XL$

(L, X) is (right) eigenpair of order k of A

similarly (L, Y) is left eigenpair if $Y^H A = L Y^H$

(and $\mathcal{R}(Y)$ is left eigenspace of A)

If X orthonormal $\rightarrow$ (L, X) orthonormal

Theo 1.4   let (L, X) be eig. pair of A.

If $U \in \mathbb{C}^{k \times k}$ nonsingular, then

$(U^{-1}LU, XU)$ is also eig. pair of A.

If $W \in \mathbb{C}^{~~k\times k~~ \, n \times n}$ nonsingular, then

$(L, W^{-1}X)$ eig. pair of $W^{-1}AW$

$AXU = XLU = XU \cdot U^{-1}LU$ etc

U represents (arbitrary) change of basis $\rightarrow$

basis for X (eigenspace) in principle not relevant (in practice it is)

$\rightarrow$ eigenvalues do not change $U^{-1}LU$ similar to L

~~[illegible]~~ $\hookrightarrow$ space determines eigenval.s not basis.

$W^{-1}AW \cdot W^{-1}X = W^{-1}AX = W^{-1}XL$

Theo 1.5

let $\mathcal{L} = \{d_1, \dots, d_k\}$ be a multisubset of eig. val.s

Then there is subspace $\mathcal{X}$ that is eig. space of A

with eigenvalues $d_1 \cdots d_k$

(Block) Schur form : $A(U_1\ U_2) = (U_1 U_2)\begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}$

where T_{11} has elements of $\mathcal{L}$ on diagonal (with mult.s. $AU_1 = U_1 T_1$ and $\mathcal{X} = \mathcal{R}(U_1)$