4 Weak convergence

In this chapter we define and discuss the notion of weak convergence in a Banach space X. Most of the results will be discussed for the particular case of $X = L^p(\mathbb{R}^d)$.
While a generic, yet quite valid, motivation for introducing weak convergence is that relaxing assumptions or requriements can in general be quite useful, there is a much deeper reason: the Banach-Alaoglu theorem. This is a deep abstract compactness result that will have a great importance in the rest of the course.
Recall that, if $(X, \|\cdot\|_X)$ is a Banach space, then its dual is

$$X^* = \{L : X \to \mathbb{C} \text{ linear and continuous}\}.$$

$$= \{L : X \to \mathbb{C} \text{ linear s.t. } \exists C > 0 \text{ s.t. } |L(x)| \leq C|x| \ \forall x \in X\}.$$

X^* is a Banach space as well, if equipped with the norm

$$\|L\|_{X^*} = \sup_{\substack{x \in X \\ \|x\|_X = 1}} |L(x)|.$$

Definition 4.1 (Weak convergence)

We say that $f_n \in X$ converges weakly to $f \in X$ ($f_n \rightharpoonup f$) as $n \to \infty$ if, $\forall L \in X^*$, $L(f_n) \to L(f)$ as $n \to \infty$.

First remark: strong convergence implies weak convergence.
Let $f_n \in X$ be strongly convergent to $f \in X$. Then, $\forall L \in X^*$,

$$|L(f_n)-L(f)| \leq \|L\|_{X^*}\|f_n-f\|_X \to 0$$

and therefore $f_n \rightharpoonup f$. The opposite implication is in general not true.

A fundamental property of convergence in $\mathbb{R}^n$ or $\mathbb{C}^n$ is that every bounded sequence has a convergent subsequence (Bolzano-Weierstrass theorem). This is what we call a compactness result. This unfortunately stops being true on generic (infinite dimensional) Banach spaces. This has to do with the fact that the unit ball in X (i.e. $\{x\in X \text{ s.t. } \|x\|=1\}$) is in general not compact with respect to the topology induced by either the strong or weak convergence.

To recover compactness we need to go to an even weaker notion of convergence.

Definition 4.2 (Weak-* convergence)

Let X^* be the dual of a Banach space X. We say that a sequence $L_n \in X^*$ converges weakly-* to $L\in X^*$ if

$$\lim_{n\to\infty} L_n(x) = L(x) \quad \forall x\in X.$$

We can thus state the announced compactness result.

Theorem 4.3 (Banach-Alaoglu theorem)

Let X be a Banach space and X^* its dual. The unit ball in X^* is compact in the weak-* topology of X^*.

Under additional assumptions on X^* (e.g. separability) this

implies that every bounded sequence has a convergent subsequence.
The theorem motivates why reflexive Banach spaces (i.e. spaces for which $X \cong X^{**}$) are particularly nice: in those spaces, X is itself the dual of the Banach space X^*, and weak and weak-$*$ topologies coincide.
We will not go further with general theory and statements, which typically occupy a functional analysis class. We will discuss all this for the concrete case of $X = L^p(\mathbb{R}^d)$.
Notice that we already know that, if $1 < p < \infty$, then

$$\left(L^p(\mathbb{R}^d)\right)^{**} \cong L^p(\mathbb{R}^d),$$

i.e. reflexivity holds. These are the cases in which we will prove a compactness result.

While all results in this chapter will be stated and discussed, for simplicity, for the Banach spaces $L^p(\mathbb{R}^d)$, (with the Lebesgue measure), most statements can be immediately generalized to $L^p(\Omega, A, \mu)$ where (Ω, A, μ) is a measure space (sometimes adding mild requirements on μ).

The first result we want to discuss is the uniqueness of the weak limit when it exists.

<u>Theorem 4.4 (Linear functionals separate points)</u>

Suppose $f \in L^p(\mathbb{R}^d)$ is such that $L(f) = 0 \ \forall L \in (L^p(\mathbb{R}^d))^*$ for $1 \le p \le \infty$. Then $f = 0$.
As a consequence, if $f_n \in L^p(\mathbb{R}^d)$ satisfies both $f_n \rightharpoonup f$ and $f_n \rightharpoonup g$ for some $f, g \in L^p(\mathbb{R}^d)$, then $f = g$.

We will prove this, an later results, using the following Lemma (which we actually have already seen and used during the course).

Lemma 4.5

Let $f \in L^p(\mathbb{R}^d)$ with $1 \le p \le \infty$. Then, if $\frac{1}{p} + \frac{1}{p'} = 1$,

$$\|f\|_{L^p} = \sup_{\|g\|_{L^{p'}(\mathbb{R}^d)} = 1} |\langle g, f\rangle| \equiv \sup_{\|g\|_{L^{p'}(\mathbb{R}^d)} = 1} \left| \int_{\mathbb{R}^d} gf \right|$$

Proof

By Hölder's inequality we have, $\forall g \in L^{p'}(\mathbb{R}^d)$ with $\|g\|_{L^{p'}} = 1$

$$|\langle g, f\rangle| \le \|f\|_{L^p} \|g\|_{L^{p'}} = \|f\|_{L^p}.$$

Taking the supremum yieds $\sup |\langle g, f\rangle| \le \|f\|_{L^p}$, which is already one direction. We now prove the opposite inequality. Let $1 \le p < \infty$ and define

$$g(x) = \begin{cases} \frac{1}{\|f\|_{L^p}^{p-1}} \overline{f(x)} \cdot |f(x)|^{p-2} & f(x) \ne 0 \\ 0 & f(x) = 0. \end{cases}$$

Then $|g|^{p'} = \frac{1}{\|f\|_{L^p}^{p'(p-1)}} |f|^{p'(p-1)} = \frac{|f|^p}{\|f\|_{L^p}^p}$, whence $g \in L^{p'}(\mathbb{R}^d)$ and $\|g\|_{L^{p'}} = 1$. We thus have

$$\langle g, f\rangle = \frac{1}{\|f\|_{L^p}^{p-1}} \cdot \int_{\mathbb{R}^d} |f|^p = \|f\|_{L^p}.$$

This concludes the proof if $1 \le p < \infty$. If instead $p = \infty$, assume then $f \ne 0$ (otherwise nothing has to be proven). Consider the set $A_\varepsilon = \{x \in \mathbb{R}^d \text{ s.t. } |f(x)| > \|f\|_\infty - \varepsilon\}$. Then $\exists B_\varepsilon \subset A_\varepsilon$ s.t. $0 < |B_\varepsilon| < A_\varepsilon$. With such a B_ε, define

$$g_\varepsilon(x) = \begin{cases} \frac{1}{|B_\varepsilon|} \frac{\overline{f(x)}}{|f(x)|} & x \in B_\varepsilon \\ 0 & x \in B_\varepsilon. \end{cases}$$

Since $|B_\varepsilon|<\infty$ and $f\in L^\infty(\mathbb{R}^d)$, we have $g_\varepsilon \in L^1(\mathbb{R}^d)$ with (5)
$\|g_\varepsilon\|_{L^1} = 1$. Then

$$\langle g_\varepsilon, f\rangle = \frac{1}{|B_\varepsilon|}\int_B |f(x)|\,dx \geq \frac{\|f\|_\infty - \varepsilon}{|B|_\varepsilon}|B|_\varepsilon = \|f\|_\infty - \varepsilon.$$

This implies $\sup_{\|g\|_{L^1}=1} |\langle g,f\rangle| \geq \|f\|_\infty$, which concludes the proof ∎

Proof of Theorem 4.4

For every $g\in L^{p'}(\mathbb{R}^d)$ we have $\langle g,f\rangle = L_g(f)=0$ by assumption. If $1\leq p<\infty$, then all elements of the dual are of the type L_g above, since $(L^p(\mathbb{R}^d))^* \cong L^{p'}(\mathbb{R}^d)$. For $p=\infty$ the dual is larger, i.e. $\exists L\in (L^\infty(\mathbb{R}^d))^*$ not of the type L_g with $g\in L^1(\mathbb{R}^d)$. But this does not play a role here. Taking the supremum over g's with $\|g\|_{L^{p'}}=1$ in the first line, we find $\|f\|_{L^p}=0$, and thus $f=0$.
To conclude, let $f_n\in L^p(\mathbb{R}^d)$ with $f_n \rightharpoonup f$, $f_n\rightharpoonup g$ and $f,g\in L^p(\mathbb{R}^d)$. By definition of weak convergence, $L(f_n)\to L(f)$ and $L(f_n)\to L(g)$, and therefore $L(f)=L(g)$ (by uniqueness of limits in $\mathbb{C}$), for all $L\in (L^p(\mathbb{R}^d))^*$. By linearity we have $L(f-g)=0$, and therefore, by the first part of the theorem, $f=g$. ∎

The next property of weak convergence that we want to discuss is the lower semicontinuity of norms: weak limits can only "lose" (decrease) norm in the limit, not gain it.

Theorem 4.6 (Weak lower semicontinuity of L^p-norms)

For $1\leq p\leq\infty$, suppose that $f_n\in L^p(\mathbb{R}^d)$ is a sequence with $f_n\rightharpoonup f\in L^p(\mathbb{R}^d)$. Then

$$\|f\|_{L^p} \leq \liminf_{n\to\infty} \|f_n\|_{L^p}.$$

Moreover, if $1<p<\infty$, and $\lim_{n\to\infty}\|f_n\|_{L^p} = \|f\|_{L^p}$, then $f_n\to f$ strongly.

Notice that $f_n \to f$ strongly implies convergence of the norms, since

$$\big| \|f_n\|_{L^p} - \|f\|_{L^p} \big| \leq \|f_n - f\|_{L^p}$$

using the triangle inequality. For the weak convergence this is not necessarily the case in general.

Proof

By Lemma 4.5 we have, for any $g \in L^{p'}(\mathbb{R}^d)$ with $\|g\|_{L^{p'}} = 1$,

$$\|f_n\|_{L^p} \geq |\langle g, f_n\rangle| = |L_g(f_n)|$$

for some $L_g \in (L^p(\mathbb{R}^d))^*$. Since $f_n \rightharpoonup f$, the right hand side converges to $L_g(f)$. The left hand side might not have a limit, but we can write

$$\liminf_{n\to\infty} \|f_n\|_{L^p} \geq |L_g(f)| = |\langle g, f\rangle|$$

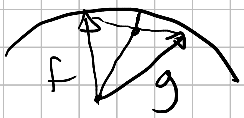
Taking the supremum over g with $\|g\|_{L^{p'}} = 1$ concludes the proof of the first statement.

Let us now show the second statement in the theorem, for $1<p<\infty$. Assume thus $f_n \rightharpoonup f$ in L^p and $\|f_n\|_{L^p} \to \|f\|_{L^p}$. We use the Hanner's inequalities. For $1 < p \leq 2$,

$$\|f+g\|_{L^p}^p + \|f-g\|_{L^p}^p \geq \left(\|f\|_{L^p} + \|g\|_{L^p}\right)^p + \big|\|f\|_{L^p} - \|g\|_{L^p}\big|^p$$

$$\left(\|f+g\|_{L^p} + \|f-g\|_{L^p}\right)^p + \big|\|f+g\|_{L^p} - \|f-g\|_{L^p}\big|^p \leq 2^p \left(\|f\|_{L^p}^p + \|g\|_{L^p}^p\right).$$

and for $2 \leq p < \infty$ the inequalities are reversed (for $p=2$ they are indeed identities (parallelogram identity)). For these, we refer to Theorem 2.5 in Lieb-Loss, or Theorem 1.4 in Benjamin Schlein's notes. These bounds imply the uniform convexity property of L^p spaces if $1<p<\infty$: for any two f, g in the unit ball of L^p with $1<p<\infty$, their middle point is well inside the ball, unless f and g are close.



$\leftarrow L^p$-ball

Using the second inequality for $1<p\leq 2$ with $f=f$, $g=f_n$, and recalling $\|f_n\|_{L^p}\to\|f\|_{L^p}$, we find ($2<p<\infty$ is left as exercise)

$$\limsup_{n\to\infty}\left[\left(\|f+f_n\|_{L^p}+\|f-f_n\|_{L^p}\right)^p+\left|\|f+f_n\|_{L^p}-\|f-f_n\|_{L^p}\right|^p\right]\leq 2^{p+1}\|f\|_{L^p}^p.$$

How about the limit of $\|f+f_n\|_{L^p}$? Clearly, $f+f_n\to 2f$, and therefore $\liminf_{n\to\infty}\|f+f_n\|_{L^p}\geq 2\|f\|_{L^p}$. However, we also have

$$\|f+f_n\|_{L^p}\leq\|f\|_{L^p}+\|f_n\|_{L^p}\xrightarrow[n\to\infty]{}2\|f\|_{L^p},$$

which implies $\limsup_{n\to\infty}\|f+f_n\|_{L^p}\leq 2\|f\|_{L^p}$, and therefore

$$\lim_{n\to\infty}\|f+f_n\|_{L^p}=2\|f\|_{L^p}.$$

We thus have an estimate of the form

$$\limsup_{n\to\infty}\left[(A_n+B_n)^p+|A_n-B_n|^p\right]\leq 2^{p+1}\|f\|_{L^p}$$

where $A_n=\|f+f_n\|_{L^p}$ and $B_n=\|f-f_n\|_{L^p}$, and $A_n\to 2\|f\|_{L^p}$. Since $x\mapsto|A+x|^p$ is strictly convex for $1<p<\infty$, we have, if $B_n\neq 0$,

$$|A_n+B_n|^p+|A_n-B_n|^p>2\left|A_n+\frac{B_n}{2}-\frac{B_n}{2}\right|^p\xrightarrow[n\to\infty]{}2\left(2\|f\|_{L^p}\right)^p=2^{p+1}\|f\|_{L^p}^p.$$

We thus find (the $>$ becomes $\geq$ in the limit)

$$2^{p+1}\|f\|_{L^p}^p\geq\limsup_{n\to\infty}\left[(A_n+B_n)^p+|A_n-B_n|^p\right]$$

$$\underset{\text{trivial}}{\geq}\liminf_{n\to\infty}\left[(A_n+B_n)^p+|A_n-B_n|^p\right]\underset{\text{ineq. above}}{\geq}2^{p+1}\|f\|_{L^p}^p.$$

This proves that $(A_n+B_n)^p + |A_n-B_n|^p \to 2^{p+1}\|f\|_{L^p}^p$, A_n also converges. But we still can't deduce convergence of A_n+B_n or A_n-B_n. Along a subsequence, we also have that $B_{n_j} \xrightarrow[j\to\infty]{} B \in \mathbb{R}^+$ (since B_n is bounded). Along n_j we can then pass to the limit and write

$$\left(2\|f\|_{L^p}+B\right)^p + \left(2\|f\|_{L^p}-B\right)^p = 2^{p+1}\|f\|_{L^p}^p.$$

But, by strict convexity,

$$\left(2\|f\|_{L^p}+B\right)^p + \left(2\|f\|_{L^p}-B\right) > 2^{p+1}\|f\|_{L^p}^p \quad \text{if } B\neq 0$$

This is a contradiction, and hence $B=0$. We have shown that B_n converges to zero along all subsequences. Hence $B_n \xrightarrow[n\to\infty]{} 0$ ∎

Another very important property of weakly convergent sequence is that they are always norm bounded (i.e., they live in some ball of finite radius inside $L^p(\mathbb{R}^d)$). This is typically one of the first ingredients of compactness arguments.

Theorem 4.7 (Uniform boundedness principle)

Let $f_n \in L^p(\mathbb{R}^d)$ be a sequence such that $\forall L \in \left(L^p(\mathbb{R}^d)\right)^*$, the sequence $L(f_n)$ is bounded in $\mathbb{C}$. Then $\exists\, C>0$ s.t. $\|f_n\|_{L^p} \leq C$ $\forall n$. As a consequence, if $f_n \rightharpoonup f$, then $\|f_n\|_{L^p} \leq C$ $\forall n$.

Proof

We focus on $1<p<\infty$ and leave the other cases as exercise. By contradiction, assume $\|f_n\|_{L^p}$ is not uniformly bounded. We can assume, up to a subsequence, that $\|f_n\|_{L^p} \geq 4^n$. Define

$$F_n := \frac{4^n}{\|f_n\|_{L^p}} f_n \in L^p(\mathbb{R}^d), \text{ with } \|F_n\|_{L^p} = 4^n.$$

Then $\forall L \in (L^p(\mathbb{R}^d))^*$, the sequences

$$L(F_n) = \frac{4^n}{\|f_n\|_{L^p}} L(f_n)$$

are certainly bounded, since $0 \le \frac{4^n}{\|f_n\|_{L^p}} \le 1$ and $|L(f_n)| \le C$.

We look for a particular $\tilde{L}$ contradicting this boundedness.
As in the proof of Lemma 4.5, the function

$$T_n = \frac{|F_n|^{p-2}\,\overline{F_n}}{\|F_n\|_{L^p}^{p-1}}$$

belongs to $L^{p'}(\mathbb{R}^d) = (L^p(\mathbb{R}^d))^*$ and $\|T_n\|_{L^{p'}} = 1$.
We also define, recursively, a sequence $\sigma_n \in \mathbb{C}$ with $|\sigma_n| = 1$ by setting $\sigma_1 = 1$ and choosing the argument (angle) of σ_n s.t.

$$\sigma_n \int T_n F_n \quad \text{and} \quad \sum_{J=1}^{n-1} 3^{-J} \sigma_J \int T_J F_n$$

have the same argument. We thus have

$$\left| \sum_{J=1}^{n} 3^{-J} \sigma_J \int_{\mathbb{R}^d} T_J F_n \right| = \left| 3^{-n} \sigma_n \int_{\mathbb{R}^d} T_n F_n + \sum_{J=1}^{n-1} 3^{-J} \sigma_J \int_{\mathbb{R}^d} T_J F_n \right|$$

$$\left(\begin{array}{c}|a+b| = |a|+|b| \text{ if} \\ a,b\in\mathbb{C} \text{ have the same} \\ \text{argument}\end{array}\right) \Rightarrow \ \ge \left| 3^{-n} \sigma_n \int_{\mathbb{R}^d} T_n F_n \right| = 3^{-n} \|F_n\|_{L^p} = \left(\frac{4}{3}\right)^n.$$

Define thus $\tilde{L} \in (L^p(\mathbb{R}^d))^*$ by setting

$$\tilde{L}(h) = \sum_{J=1}^{\infty} 3^{-J} \sigma_J \int_{\mathbb{R}^d} T_J h.$$

Since, by the triangle and Hölder inequalities,

$$|\tilde{L}(h)| \le \sum_{J=1}^{\infty} 3^{-J} \left| \int_{\mathbb{R}^d} T_J h \right| \le \|T_J\|_{L^{p'}} \, \|h\|_{L^p} \sum_{J=1}^{\infty} 3^{-J}$$

$$= \frac{1}{2} \|h\|_p,$$

we deduce that $\tilde{L} \in (L^p(\mathbb{R}^d))^*$, and, by definition of norm in the dual,

$$\|\tilde{L}\|_{(L^p(\mathbb{R}^d))^*} \le \frac{1}{2}.$$

We obtain a contradiction by showing $\tilde{L}(F_n) \to \infty$.
Indeed,

$$|\tilde{L}(F_n)| = \left| \sum_{J=1}^{\infty} 3^{-J} \sigma_J \int_{\mathbb{R}^d} T_J F_n \right|$$

$$\geq \left| \sum_{J=1}^{n} 3^{-J} \sigma_J \int T_J F_n \right| - \sum_{J=n+1}^{\infty} 3^{-J} \|T_J\|_{L^{p'}} \|F_n\|_{L^p}$$

$|A+B| \geq |A| - |B|$

$$\geq \left(\frac{4}{3}\right)^n - \sum_{J=n+1}^{\infty} 3^{-J} 4^n = \frac{1}{2}\left(\frac{4}{3}\right)^n \to \infty$$

$\sum_{J=0}^{N} \alpha^{-J} = \frac{1-\alpha^{-(N+1)}}{1-\alpha^{-1}} \quad \alpha > 1.$

This proves a contradiction. ∎

Let us recall that we are always aiming towards the Banach-Alaoglu theorem for L^p-spaces, i.e. a compactness result for the notion of weak convergence.
To this end, we will need the notion of separability, which gives a limitation on the size of certain normed or metric spaces (it extends more in general to topological spaces).

Definition 4.8 (Separable space)

A normed vector space $(X, \|\cdot\|_X)$ is called separable if $\exists B \subset X$ countable that is dense in X, i.e., $\forall \varepsilon > 0$, $\forall f \in X$, $\exists g \in B$ s.t.

$$\|g - f\|_X < \varepsilon.$$

Theorem 4.9 (Separability of L^p)

For $1 \leq p < \infty$, the space $L^p(\mathbb{R}^d)$ is separable.
$L^\infty(\mathbb{R}^d)$, in turn, is not separable.

We will not see the proof of Theorem 5.9 in full detail, but we explain which one is the explicit choice of the countable dense set.

Define the set of cubes, for $J \in \mathbb{N}$ and $m = (m_1, \dots, m_d) \in \mathbb{Z}^d$,

$$\Gamma_{J,m} = \left\{ x \in \mathbb{R}^d, \quad \frac{m_i}{2^J} \leq x_i \leq \frac{m_i + 1}{2^J} \quad \text{for } i = 1, \dots, d \right\}$$

These have side 2^{-J} and one vertex at $\frac{m}{2^J} \in \mathbb{R}^d$. As m, J vary, they cover the whole $\mathbb{R}^d$. Notice that since $\mathbb{N}$ and $\mathbb{Z}^d$ are countable, $\{\Gamma_{J,m}\}$ is countable.

Define then the family of functions at fixed J.

$$\tilde{\mathcal{F}}_J = \Big\{ f : \mathbb{R}^n \to \mathbb{C} \text{ s.t. } \operatorname{supp} f \subseteq \bigcup_{k=1}^{N} \Gamma_{J,m_k} \text{ for some } \Gamma_{J,m_k} \text{ and some finite } N$$
$$\underline{\underline{\text{and}}} \quad f(x) = C_{J,k} \in \mathbb{Q} \quad \forall x \in \Gamma_{J,m_k} \Big\}$$

This is also a countable family: it is parametrized by finite subsets of the set of cubes (and the set of finite subsets of a countable set is countable) and by $\mathbb{Q}$, which is countable.

We finally take $\mathcal{F} = \bigcup_{J \in \mathbb{N}} \tilde{\mathcal{F}}_J$. This is the set of functions with support on finitely many cubes, and with rational values.

Intuitively, the construction works since:

- the cubes can cover arbitrarily well $\mathbb{R}^d$;
- the values of the functions (in $\mathbb{Q}$) approximate arbitrarily well real values;
- simple functions are nicely dense in L^p (Theorem 0.5).

There are many ways of trying to understand why this construction cannot work for L^∞. One reason is that the L^∞-norm responds very badly to small perturbations: two functions that differ by a finite quantity on an arbitrarily small set are very far in L^∞.

More deeply, the lack of separability of L^∞ is due to the fact that there are "too many" disjoint unit balls. On a very nice separable space like, e.g., $\mathbb{R}$, one can have only countably many mutually disjoint unit balls. This is not true in L^∞: any two characteristic functions of measurable set which differ by a set of positive measure, no matter how small, have L^∞ distance equal to one.

We are finally ready for the Banach-Alaoglu theorem.

Theorem 4.10 (Banach-Alaoglu, L^p version)

Let $f_n \in L^p(\mathbb{R}^d)$ with $1<p<\infty$ be a bounded sequence, i.e., $\|f_n\|_{L^p} \leq C$ for some fixed C. Then $\exists$ a subsequence f_{n_J} and $\exists f \in L^p(\mathbb{R}^d)$ such that $f_{n_J} \rightharpoonup f$ as $J \to \infty$.

Proof

We look for f_{n_J} such that $L(f_{n_J})$ converges $\forall L \in (L^p(\mathbb{R}^d))^*$, i.e. such that $\int_{\mathbb{R}^d} f_{n_J} g$ converges $\forall g \in L^{p'}(\mathbb{R}^d)$.

We know that $L^{p'}(\mathbb{R}^d)$ is separable, by Theorem 4.9, which means there exists a countable dense subset $\{\varphi_\ell, \ell \in \mathbb{N}\} \subset L^{p'}(\mathbb{R}^d)$.

Let's look for f_{n_J} s.t. $\int f_{n_J} \varphi_\ell$ converges as $J \to \infty$ $\forall \ell \in \mathbb{N}$.

We will then argue by density.

We first build a sequence of complex numbers at fixed J: define

$$\alpha_J^1 := \int_{\mathbb{R}^d} f_J \phi_1 .$$

By Hölder, $|\alpha_J^1| \leq \|\phi_1\|_{L^{p'}} \|f_J\|_{L^p} \leq C$ uniformly in J, and therefore $\exists n_{1,J}$ subsequence s.t. $\alpha^1_{n_{1,J}}$ converges in $\mathbb{C}$ as $J \to \infty$. Define now

$$\alpha_J^2 := \int_{\mathbb{R}^d} f_J \phi_2 .$$

Then $|\alpha_J^2| \leq C$, and in particular also $|\alpha^2_{n_{1,J}}| \leq C$. Then $\exists n_{2,J}$ subsequence of $n_{1,J}$ s.t. $\alpha^2_{n_{2,J}}$ and $\alpha^1_{n_{2,J}}$ converge in $\mathbb{C}$ as $J \to \infty$.

We argue recursively. $\forall K \in \mathbb{N}$ we find $n_{K,J}$ s.t.

$$\alpha^{m}_{n_{K,J}} := \int_{\mathbb{R}^d} f_{n_{K,J}} \varphi_m$$

converges as $J \to \infty$ for all $m = 1, 2 \dots K$.
We now consider the subsequence $n_{K,K}$ for f_J.
Let $\ell \in \mathbb{N}$. For $K \geq \ell$, $n_{K,K}$ is a subsequence of $n_{\ell,K}$ by the construction above. This implies that

$$\lim_{K \to \infty} \int f_{n_{K,K}} \varphi_\ell \, d\lambda_n$$

exists for all $\ell \in \mathbb{N}$. This is Known as Cantor's diagonal argument.
We now need to use a density argument to replace φ_ℓ above by $g \in L^{p'}(\mathbb{R}^d)$. Fix then $g \in L^{p'}(\mathbb{R}^d)$. We want to prove that $\int_{\mathbb{R}^d} f_{n_{K,K}} g$ is a Cauchy sequence.

Let us use the density property of the $\{\varphi_\ell\}$'s. $\forall \varepsilon > 0 \ \exists \ell$ s.t. $\|g - \varphi_\ell\|_{L^{p'}} < \varepsilon$ (by Theorem 4.9). Then

$$\left| \int_{\mathbb{R}^d} g f_{n_{K,K}} - \int_{\mathbb{R}^d} g f_{n_{i,i}} \right| \leq \int_{\mathbb{R}^d} |g - \phi_\ell| \, |f_{n_{K,K}}| + \left| \int_{\mathbb{R}^d} \phi_\ell \left(f_{n_{K,K}} - f_{n_{i,i}} \right) \right|$$
$$+ \int |g - \phi_\ell| \, |f_{n_{i,i}}|$$

$$\leq \|g - \phi_\ell\|_{L^{p'}} \left(\|f_{n_{K,K}}\|_{L^p} + \|f_{n_{i,i}}\|_{L^p} \right)$$
$$+ \left| \int_{\mathbb{R}^d} \phi_\ell \left(f_{n_{K,K}} - f_{n_{i,i}} \right) \right|$$

$$\leq 2\varepsilon C + \left| \int_{\mathbb{R}^d} \phi_\ell \left(f_{n_{K,K}} - f_{n_{i,i}} \right) \right| \leq C\varepsilon$$

for i, K large enough, since $\int \phi_\ell f_{n_{K,K}}$ is convergent and hence

Cauchy. This shows that $\int g f_{n_{K,K}} \equiv L_g(f_{n_{K,K}})$ converges as $K \to \infty$ $\forall g \in L^{p'}(\mathbb{R}^d)$.
We have to conclude by showing that the knowledge of convergence of $L_g(f_{n_{K,K}})$ allows to find an $f \in L^p(\mathbb{R}^d)$ s.t. $f_{n_{K,K}} \rightharpoonup f$.
For this we use the fact that $(L^{p'}(\mathbb{R}^d))^* \cong L^p(\mathbb{R}^d)$, i.e. $L^p(\mathbb{R}^d)$ is reflexive. Define the functional on $L^{p'}(\mathbb{R}^d)$

$$H(g) := \lim_{K\to\infty} L_g(f_{n_{K,K}}) \equiv \lim_{K\to\infty} \int g\, f_{n_{K,K}}$$

Linearity of H is clear. Boundedness follows from

$$\left| \int_{\mathbb{R}^d} g\, f_{n_{K,K}} \right| \leq C \|g\|_{L^{p'}} \cdot \| f_{n_{K,K}} \|_{L^p} \leq C \|g\|_{L^{p'}}$$

uniformly in K: it implies $|H(g)| \leq C\|g\|_{L^{p'}}$, i.e. boundedness.
Hence $H \in (L^{p'}(\mathbb{R}^d))^* \cong L^p(\mathbb{R}^d)$, and therefore $\exists f \in L^p(\mathbb{R}^d)$ s.t.

$$\int_{\mathbb{R}^d} fg = H(g) = \lim_{K\to\infty} \int_{\mathbb{R}^d} f_{n_{K,K}}\, g \qquad \forall g \in L^{p'}(\mathbb{R}^d).$$

The left and right side of these equalities are precisely the definition of $f_{n_{K,K}} \rightharpoonup f$. ■

As follows already from the formulation in the introduction, this theorem can be extended to all reflexive Banach spaces, i.e., those for which $X^{**} \cong X$.

The last topic of this section concernes weak convergence from the point of view of Sobolev spaces, and the so-called compact Sobolev embeddings.
Before going there, it is worth spending some time discussing how the results presented so far in this section apply to Sobolev spaces (which are particular Banach spaces).

First question: what is the dual space of $H^1(\mathbb{R}^d)$?
By the Fourier characterization of $H^1(\mathbb{R}^d)$ (Theorem 3.9), we know

$$H^1(\mathbb{R}^d) = \left\{ f \in L^2(\mathbb{R}^d) \mid k \mapsto |k| \hat{f}(k) \text{ belongs to } L^2(\mathbb{R}^d) \right\}$$

and

$$\|f\|_{H^1}^2 = \int_{\mathbb{R}^d} |\hat{f}(k)|^2 (1 + 4\pi^2 |k|^2)\, dk .$$

This shows that $H^1(\mathbb{R}^d)$ is (isometrically isomorphic to) $L^2(\mathbb{R}^d, d\mu)$, where the measure μ is

$$d\mu(k) = (1 + 4\pi^2 |x|^2)\, dk .$$

This means in particular that the dual of $H^1(\mathbb{R}^d) \cong L^2(\mathbb{R}^d, d\mu)$ is isomorphic to $H^1(\mathbb{R}^d)$ itself, and thus for all $L \in (H^1(\mathbb{R}^d))^*$, $L = L_g$ for some $g \in H^1(\mathbb{R}^d)$ and

$$L_g(f) = \int_{\mathbb{R}^d} \hat{g}(k) \hat{f}(k) (1 + 4\pi^2 |k|^2)\, dk .$$

Thus, a sequence f_n in $H^1(\mathbb{R}^d)$ converges weakly to $f \in H^1(\mathbb{R}^d)$ ($f_n \rightharpoonup f$) if

$$\lim_{n \to \infty} \int_{\mathbb{R}^d} \hat{g}(k) \left(\hat{f}_n(k) - \hat{f}(k) \right) (1 + 4\pi^2 |k|^2)\, dk = 0 \qquad \forall g \in H^1(\mathbb{R}^d),$$

or, in other words,

$$\lim_{n \to \infty} \left[\int_{\mathbb{R}^d} g (f_n - f) + \int_{\mathbb{R}^d} \nabla g (\nabla f_n - \nabla f) \right] = 0 \qquad \forall g \in H^1(\mathbb{R}^d).$$

Notice that this representation of the dual is the "Hilbert space" one, i.e. essentially $(H^1(\mathbb{R}^d))^* \cong H^1(\mathbb{R}^d)$. There is a different possible representation: for $g_0, g_1, \ldots, g_d \in L^2(\mathbb{R}^d)$, the linear map

$$L(f) = \int_{\mathbb{R}^d} g f + \sum_{j=1}^{d} \int_{\mathbb{R}^d} g_j \partial_j f \qquad \forall f \in H^1(\mathbb{R}^d)$$

defines a continuous linear functional on $H^1(\mathbb{R}^d)$.
Suitably choosing the g's, one can see that $f_n \underset{n\to\infty}{\rightharpoonup} f$ in $H^1(\mathbb{R}^d)$ if and only if

$$f_n \underset{n\to\infty}{\rightharpoonup} f \text{ in } L^2(\mathbb{R}^d) \quad \underline{\underline{\text{and}}} \quad \nabla f_n \underset{n\to\infty}{\rightharpoonup} \nabla f \text{ in } L^2(\mathbb{R}^d).$$

We can then argue quite easily that the Banach-Alaoglu Theorem in the form of Theorem 4.10 must hold for $H^1(\mathbb{R}^d)$ as well.

Proposition 4.11

Let f_n be a bounded sequence in $H^1(\mathbb{R}^d)$. Then there exists a subsequence f_{n_j} s.t. $f_{n_j} \rightharpoonup f \in H^1(\mathbb{R}^d)$ weakly in $H^1(\mathbb{R}^d)$.

Proof

By definition of H^1-norm, $\|f_n\|_{H^1}$ bounded means $\|f_n\|_{L^2}, \|\nabla f_n\|_{L^2}$ bounded. But then, by Banach-Alaoglu for $L^2(\mathbb{R}^d)$, $\exists$ a subsequence f_{n_j} s.t. $f_{n_j} \rightharpoonup f \in L^2(\mathbb{R}^d)$ weakly in $L^2(\mathbb{R}^d)$. Similarly, (we omit extracting further subsequences) $\exists\, g_1, \dots g_d \in L^2(\mathbb{R}^d)$ such that $\partial_j f_{n_j} \rightharpoonup g_j$ weakly in $L^2(\mathbb{R}^d)$. If we can show $g_j = \partial_j f$ then we are done. Let $\phi \in \mathcal{D}(\mathbb{R}^d)$ be a test function. Then

$$\partial_j T_f(\phi) \underset{f\in L^1_{loc}}{=} -\int_{\mathbb{R}^d} f\, \partial_j \phi \underset{f_n \rightharpoonup f}{=} -\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n\, \partial_j \phi = -\lim_{n\to\infty} T_{f_n}(\partial_j \phi)$$

$$\underset{\nabla f_n \in L^1_{loc}}{=} \lim_{n\to\infty} \int_{\mathbb{R}^d} (\partial_j f_n)\, \phi = \int_{\mathbb{R}^d} g_j\, \phi.$$

This proves $\partial_j f = g_j \in \mathcal{D}'(\mathbb{R}^d)$, but $g_j \in L^2(\mathbb{R}^d)$ by construction. ∎

We are finally ready to go towards compact Sobolev embeddings. We recall that, if $\Omega \subseteq \mathbb{R}^d$ is nice enough, then

$$\|f\|_{L^q(\Omega)} \le C \|\nabla f\|_{L^2(\Omega)} \quad \text{if } d \geq 3 \text{ and } q = \frac{2d}{d-2},$$

and more in general $\|f\|_{L^p} \le C\|f\|_{H^1}$ for all $p\in[2,\frac{2d}{d-2}]$.

Now, if instead of $\mathbb{R}^d$ we are on $\Omega\subset\mathbb{R}^d$ bounded and nice enough, then one can show that the embedding $H^1(\Omega)\hookrightarrow L^p(\Omega)$ (for $p<\frac{2d}{d-2}$) is compact, i.e. the unit ball of $H^1(\Omega)$ is a compact subset of $L^p(\Omega)$. In other words, every bounded sequence in $H^1(\Omega)$ has a subsequence which converges strongly in $L^p(\Omega)$.

We state the next result in a slightly different manner.

Theorem 4.12 (Weak convergence implies strong convergence on small sets)

Let $f_n\in H^1(\mathbb{R}^d)$ be a weakly convergent sequence in $H^1(\mathbb{R}^d)$, and $f\in H^1(\mathbb{R}^d)$ its limit. Let A be a set of finite measure and χ_A its characteristic function. Then $\chi_A f_n \xrightarrow[n\to\infty]{} \chi_A f$ strongly in L^q for $1\le q<\frac{2d}{d-2}$ if $d\geq 3$, and for $1\le q<\infty$ if $d=1,2$. For $d=1$ the convergence is pointwise, and uniform on every compact set of $\mathbb{R}^d$.

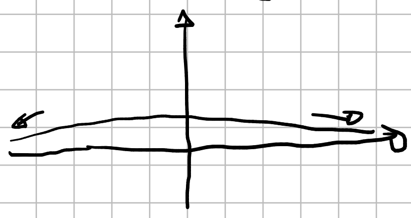
The critical case $q=\frac{2d}{d-2}$ has an explicit counterexample, due to the fact that $\|f\|_{L^{\frac{2d}{d-2}}}$ and $\|\nabla f\|_{H^1}$ scale similarly. For $g\in C_c^\infty(\mathbb{R}^d)$, the sequence $g_n(x)=n^{d/q}g(nx)$ with $q=\frac{2d}{d-2}$ satisfies $\|g_n\|_{L^q}=\|g\|_{L^q}$ and $\|\nabla g_n\|_{L^2}=\|\nabla g\|_{L^2}$; and also $g_n\rightharpoonup 0$ weakly in L^q and H^1. But it has no hope of converging strongly to zero in L^q, since its norm is constant.

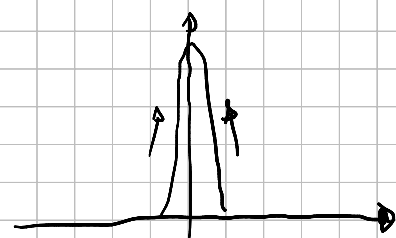
Before proving the result, it is worth discussing in some detail what it means. We are considering a weakly convergent sequence $f_n\rightharpoonup f$ in $H^1(\mathbb{R}^d)$ (i.e. $f_n-f\rightharpoonup 0$ and $\nabla f_n-\nabla f\rightharpoonup 0$ in L^2). When does a sequence $g_n\rightharpoonup 0$ in $H^1(\mathbb{R}^d)$ have a chance of converging strongly in $L^2(\mathbb{R}^d)$ (or $L^q(\mathbb{R}^d)$ with $1\le q<\frac{2d}{d-2}$ as in the Theorem)?

It turns out that weakly convergent sequences (and, more in general, bounded sequences) in $H^1(\mathbb{R}^d)$ do not show an unlimited variety of behaviors.

Let us discuss the following four typical phenomena (see also exercise 3, sheet 5).

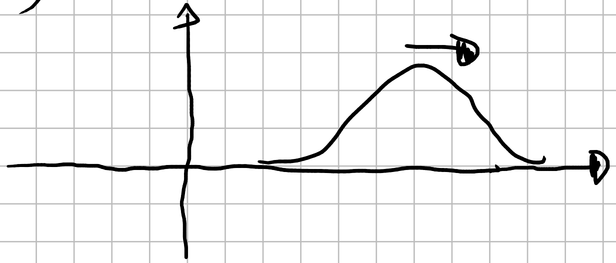
1) vanishing (melting snowman): $g_n(x) = n^{-\frac{d}{2}} g\left(\frac{x}{n}\right)$ $g \in C_c^\infty(\mathbb{R}^d)$



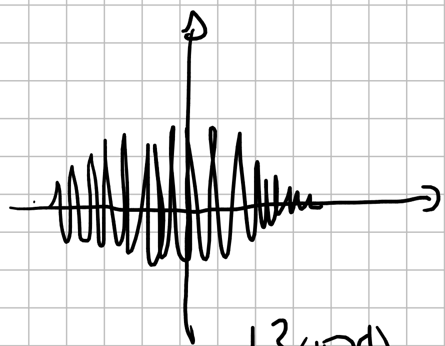


2) concentration (blow-up): $g_n(x) = n^{\frac{d}{2}} g(nx)$

3) translation (escape to infinity): $g_n(x) = g(x - nv)$ $v \in \mathbb{R}^d$



4) oscillations: $g_n(x) = e^{i n v \cdot x} g(x)$



All the above sequences satisfy $g_n \rightharpoonup 0$ in $L^2(\mathbb{R}^d)$ and $\|g_n\|_{L^2} = \|g\|_{L^2} = \text{const}$. As a consequence, none of them converge strongly.

Notice that the four examples above are pairwise "dual to each other" under a Fourier transform, in the following sense: if g_n is of the type 1), then $\hat{g}_n$ is of the type 2), and vice-versa. Analogously, if g_n is of the type 3), then $\hat{g}_n$ is of the type 4), and vice-versa (recall that the FT maps translations to complex phases and vice-versa).

There exists a beautiful theory (called by some "concentration-compactness") which allows to say many things about sequences in $H^1(\mathbb{R}^d)$, in terms of 1)–4). This is quite sophisticated, and we will not discuss it in detail at all (it was developed,

among others, by Brezis, Lieb, Lions, .. in the 80's). We will discuss some of its consequences.
Recall that, in Theorem 4.12, we are dealing with a bounded (in fact, weakly convergent) sequence in $H^1(\mathbb{R}^d)$.
This allows to say, at least heuristically, that, among the "generic" behaviors 1) - 4), we cannot have 2) and 4), since $\|\nabla g_n\|_{L^2}$ is not bounded in those cases.
In fact, it can be proven that a generic bounded sequence in $H^1(\mathbb{R}^d)$ is made of "building blocks" of the type 1) and 3), in the sense that it is (up to a subsequence) a linear combinations of pieces that retain their form, and possibly escape to ∞ (type 3)), plus a reminder that vanishes in the sense of 1). Moreover, the pieces that retain their shape, and might escape to ∞, converge strongly in L^q with $2 \leq q < \frac{2d}{d-2}$ once "translated back". In other words, something like

$$\lim_{K \to \infty} \left\| f_n - \sum_{J=1}^{K} \varphi_n^{(J)}(\cdot - x_n^{(J)}) - \psi_n \right\|_{H^1} = 0,$$

where $x_n^{(J)}$ are translations, $\varphi_n^{(J)} \to \varphi^{(J)}$ strongly in $L^q(\mathbb{R}^d)$ for $2 \leq q < \frac{2d}{d-2}$ and for some $\varphi^{(J)} \in L^q(\mathbb{R}^d)$, and ψ_n "vanishes" as in 1).

With this in mind, let us go back to Theorem 4.12. Here, we are killing the possibility of having something going to infinity, since we are multiplying by χ_A, with bounded A. With respect to the above description then, we don't even need to "translate back" the sequence. It is then quite natural to directly expect strong convergence, and this is precisely the content of the theorem.

Our ultimate aim is now to prove Theorem 4.12. The proof will use a relatively common tool called the heat kernel. Since we haven't discussed it in detail in this course yet, it is worth spending some time on it.

Definition 4.13 (Heat Kernel)

For any $t>0$ define the function $e^{t\Delta}: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$

$$e^{t\Delta}(x,y) := (4\pi t)^{-d/2} e^{-\frac{|x-y|^2}{4t}}.$$

We interpret this as the integral kernel of a linear operator acting of $f \in L^2(\mathbb{R}^d)$ by

$$(e^{t\Delta}f)(x) := \int_{\mathbb{R}^d} e^{t\Delta}(x,y)\, f(y)\, dy = \frac{1}{(4\pi t)^{d/2}} \int e^{-\frac{|x-y|^2}{4t}} f(y) dy$$

Recalling Young's inequality $\|f*g\|_{L^2} \leq \|f\|_{L^1} \|g\|_{L^2}$ it is straight forward to see that, at the very least, $e^{t\Delta}f \in L^2(\mathbb{R}^d)$.

Using the properties of the Fourier transform (Exercise #2 sheet #1 and Theorem 1.9) we see that

$$\widehat{(e^{t\Delta}f)}(k) = e^{-4\pi^2|k|^2 t}\, \hat{f}(k)$$

This explains the notation $e^{t\Delta}$, since the Laplace operator Δ acts as multiplication by $-|k|^2$ after Fourier transform.

You will see in exercise classes that, not only $e^{t\Delta}f \in L^2(\mathbb{R}^d)$, but it is in fact infinitely differentiable and it satisfies the heat equation

$$\begin{cases} \partial_t (e^{t\Delta}f) = \Delta(e^{t\Delta}f) \\ \lim\limits_{t\to 0^+} e^{t\Delta}f = f \quad \text{in } L^2(\mathbb{R}^d). \end{cases}$$

The heat kernel can be thought of as an infinitely regularizing operator, and one that approximates an identity: for any $t>0$, no matter how small, $e^{t\Delta}f$ is immediately infinitely smooth, and as $t\to 0$

it converges to f in $L^2(\mathbb{R}^d)$. We have of course already seen this, since it is a particular case of mollification.
It turns out that the heat kernel also gives the following nice characterization of functions in $H^1(\mathbb{R}^d)$.

Theorem 4.14

Let $f \in L^2(\mathbb{R}^d)$. Then $f \in H^1(\mathbb{R}^d)$ if and only if the quantity

$$I_t(f) := \frac{1}{t}\left[\langle f, f\rangle - \langle f, e^{t\Delta} f\rangle\right]$$

is uniformly bounded for $t > 0$. In this case we have

$$\sup_{t>0} I_t(f) = \lim_{t\to 0^+} I_t(f) = \|\nabla f\|_2^2 .$$

To understand the result one can think that, if a "Taylor series" in the square bracket was possible, it would yield

$$I_t(f) \text{ "}\simeq\text{" } \frac{1}{t}\left[\langle f, f\rangle - \langle f, f\rangle - t\langle f, \Delta f\rangle\right]$$

$$= -\langle f, \Delta f\rangle = \langle \nabla f, \nabla f\rangle = \|\nabla f\|_{L^2}^2 .$$

This is of course not rigorously justified here (e.g., $\Delta f \notin L^2(\mathbb{R}^d)$ in general).

Proof

By the Fourier characterization of $H^1(\mathbb{R}^d)$, we can just show that $I_t(f)$ is uniformly bounded if and only if

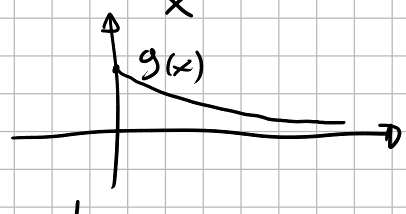
$$\int |\hat{f}(k)|^2 (1 + 4\pi^2 |k|^2)\, dk < \infty .$$

Observe now that, by the above property $\widehat{e^{t\Delta} f}(k) = e^{-4\pi^2|k|^2 t} \hat{f}(k)$ and by Plancherel, we can rewrite $I_t(f)$ as

$$I_t(f) = \frac{1}{t}\int_{\mathbb{R}^d} \left[1 - e^{-4\pi^2|k|^2 t}\right] |\hat{f}(k)|^2\, dk$$

$$= \int_{\mathbb{R}^d} \frac{1 - e^{-4\pi^2 |K|^2 t}}{4\pi^2 |K|^2 t} 4\pi^2 |K|^2 |\hat{f}(K)|^2 \, dK.$$

One can now check that the function $g(x) = \dfrac{1 - e^{-x}}{x}$ is decreasing for $x > 0$, with $\lim_{x \to 0^+} g(x) = 1$.



We then want to pass to the $t \to 0$ limit using monotone convergence. First, we write

$$\sup_{t > 0} I_t(f) = \lim_{t \to 0^+} \int_{\mathbb{R}^d} \frac{1 - e^{-4\pi^2 |K|^2 t}}{4\pi^2 |K|^2 t} 4\pi^2 |K|^2 |\hat{f}(K)|^2 \, dK,$$

since the integral grows when t decreases. We then continue using monotone convergence

$$\sup_{t > 0} I_t(f) = \lim_{t \to 0^+} \int_{\mathbb{R}^d} \frac{1 - e^{-4\pi^2 |K|^2 t}}{4\pi^2 |K|^2 t} 4\pi^2 |K|^2 |\hat{f}(K)|^2 \, dK,$$

$$= \int_{\mathbb{R}^d} 4\pi |K|^2 |\hat{f}(K)|^2 \, dK = \|\nabla f\|_{L^2}^2.$$

Monotone convergence doesn't ensure that the limit is finite. Still, any of the two sides is finite if and only if the other is, which is precisely the statement we wanted to prove. ∎

We are finally ready to prove Theorem 4.12. Let us recall the statement.

Theorem 4.12 (Weak convergence implies strong convergence on small sets)

Let $f_n \in H^1(\mathbb{R}^d)$ be a weakly convergent sequence in $H^1(\mathbb{R}^d)$, and $f \in H^1(\mathbb{R}^d)$ its limit. Let A be a set of finite measure and χ_A its characteristic function. Then

$\chi_A f_n \xrightarrow[n\to\infty]{} \chi_A f$ strongly in L^q for $1 \leq q < \frac{2d}{d-2}$ if $d \geq 3$, and for $1 \leq q < \infty$ if $d = 1, 2$. For $n=1$ the convergence is pointwise, and uniform on every compact set of $\mathbb{R}^d$.

Proof

We divide the proof into separate main steps

1. Approximation through the heat Kernel.

$\forall g \in H^1(\mathbb{R}^d)$ and $t > 0$ we claim that the following estimate holds true

$$\| g - e^{t\Delta} g \|_{L^2} \leq \| \nabla g \|_{L^2} \sqrt{t}.$$

Indeed, recalling that

$$\widehat{(e^{t\Delta} g)}(K) = e^{-4\pi^2 |K|^2 t} \, \hat{g}(K)$$

and using Plancherel, we find

$$\| g - e^{t\Delta} g \|_{L^2}^2 = \int_{\mathbb{R}^d} |\hat{g}(K)|^2 \left(1 - e^{-4\pi^2 t |K|^2}\right)^2 dK$$

$$\leq \int_{\mathbb{R}^d} |\hat{g}(K)|^2 \, 4\pi^2 t \, |K|^2 \, dK = t \, \| \nabla g \|_{L^2}.$$

where we have used $|1 - e^{-x^2}| \leq 1$ for one factor and $|1 - e^{-x^2}| \leq x^2$ for the other one.

2. Strong L^2-convergence for the approximated sequence.

We want to show that, for every $t > 0$, we have

$$\| \chi_A (e^{t\Delta} f_n - e^{t\Delta} f) \|_{L^2} \to 0 \quad \text{as} \quad n \to \infty.$$

To show this, recall that, by definition of heat Kernel,

$$e^{t\Delta} f_n(x) - e^{t\Delta} f(x) = \int_{\mathbb{R}^d} \frac{e^{-|x-y|^2/4t}}{(4\pi t)^{d/2}} \left(f_n(y) - f(y) \right) dy,$$

and the left hand side converges to zero as $n\to\infty$ $\forall x\in\mathbb{R}^d$, since $f_n\rightharpoonup f$ weakly in $L^2(\mathbb{R}^d)$ and the function

$$y\longmapsto \frac{e^{-|x-y|^2/4t}}{(4\pi t)^{d/2}}$$

belongs to $L^2(\mathbb{R}^d)$ for every $x\in\mathbb{R}^d$.

This implies immediately that $\chi_A(x)\,(e^{t\Delta}f_n)(x)\to\chi_A(x)\,(e^{t\Delta}f)(x)$ as $n\to\infty$ pointwise for all $x\in\mathbb{R}^d$.

We want to lift this to convergence in $L^2(\mathbb{R}^d)$ using dominated convergence. To this end, recall first that, $\forall g\in L^2(\mathbb{R}^d)$,

$$|e^{t\Delta}g(x)|\leq\int_{\mathbb{R}^d}\frac{e^{-|x-y|^2/4t}}{(4\pi t)^{d/2}}\,|g(y)|\,dy\leq C_t\,\|g\|_{L^2},$$

which follows for example from Young's inequality $\|h*g\|_\infty\leq\|h\|_{L^2}\,\|g\|_{L^2}$. C_t diverges as $t\to 0$, but for the moment t is fixed.

To apply dominated convergence for $\chi_A e^{t\Delta}f_n\xrightarrow{n\to\infty}\chi_A e^{t\Delta}f$ we want to control $|\chi_A e^{t\Delta}f_n-\chi_A e^{t\Delta}f|^2$ with a fixed L^1 function.

We write

$$\left|\chi_A(x)\,(e^{t\Delta}f_n)(x)-\chi_A(x)\,(e^{t\Delta}f)(x)\right|\leq C_t\,\chi_A(x)\left(\|f_n\|_{L^2}+\|f\|_{L^2}\right)$$

$$\leq C_t\,\chi_A(x)$$

where the last step follows since $f_n\rightharpoonup f$ implies that $\|f_n\|_{L^2}\leq C$ by the uniform boundedness principle. Since $\chi_A\in L^2(\mathbb{R}^d)$ (recall that A has finite measure by assumption), we can indeed use dominated convergence and argue that

$$\|\chi_A\,(e^{t\Delta}f_n-e^{t\Delta}f)\|^2=\int_{\mathbb{R}^d}|\chi_A\,(e^{t\Delta}f_n)-\chi_A\,(e^{t\Delta}f)|^2\xrightarrow[n\to\infty]{}0.$$

③ Strong L^2-convergence.

We now show that $\|\chi_A f_n - \chi_A f\|_{L^2} \to 0$ as $n\to\infty$.
To show this we write

$$\|\chi_A f_n - \chi_A f\|_{L^2} \le \|\chi_A(f_n - e^{t\Delta}f_n)\|_{L^2} + \|\chi_A(e^{t\Delta}f_n - e^{t\Delta}f)\|_{L^2} + \|\chi_A(e^{t\Delta}f - f)\|_{L^2}$$

$$\le \sqrt{t}\,\|\nabla f_n\|_{L^2} + \|\chi_A(e^{t\Delta}f_n - e^{t\Delta}f)\|_{L^2} + \sqrt{t}\,\|\nabla f\|_{L^2}$$

$$\le C\sqrt{t} + \|\chi_A(e^{t\Delta}f_n - e^{t\Delta}f)\|_{L^2},$$

where the second inequality follows from step ① and the third one from the uniform boundedness principle.
Let us fix $\varepsilon>0$, and accordingly fix $t>0$ s.t. $C\sqrt{t}\le \frac{\varepsilon}{2}$.
For such a t, from step ② we find $n_0\in\mathbb{N}$ such that

$$\|\chi_A(e^{t\Delta}f_n - e^{t\Delta}f)\|_{L^2} \le \frac{\varepsilon}{2} \qquad \forall n\ge n_0.$$

This implies

$$\|\chi_A(f_n - f)\|_{L^2} \le \varepsilon \qquad \forall n\ge n_0,$$

which is precisely $\chi_A f_n \to \chi_A f$ strongly in $L^2(\mathbb{R}^d)$ as $n\to\infty$.

④ Strong L^q-convergence.

We now show $\|\chi_A(f_n - f)\|_{L^q} \to 0$ $n\to\infty$ for $1\le q<\frac{2d}{d-2}$ if $d\ge 3$ and $1\le q<\infty$ if $d=1,2$.
Assume first $1\le q\le 2$. Then, by Hölder (and $\chi_A^q = \chi_A = \chi_A^2$)

$$\|\chi_A(f_n - f)\|_{L^q} = \left[\int_{\mathbb{R}^d} \chi_A\cdot\chi_A|f_n-f|^q\right]^{\frac{1}{q}} \le \|\chi_A\|_{L^{\alpha'}}^{\frac{1}{q}}\,\|\chi_A(f_n-f)^q\|_{L^\alpha}^{\frac{1}{q}}$$

$$= \|\chi_A\|_{L^{\alpha'}}^{\frac{1}{q}} \left(\int \chi_A|f_n-f|^{q\alpha}\right)^{\frac{1}{q\alpha}}.$$

Imposing $q\alpha = 2$ we get $\frac{1}{\alpha'} = 1 - \frac{1}{\alpha} = 1 - \frac{q}{2}$, and therefore

$$\| \chi_A (f_n - f) \|_{L^q} \le \left(\int \chi_A \right)^{\frac{1}{q\alpha'}} \cdot \left(\int \chi_A |f_n - f|^2 \right)^{\frac{1}{2}}$$

$$= \| \chi_A \|_{L^1}^{\frac{1}{q} - \frac{1}{2}} \, \| \chi_A (f_n - f) \|_{L^2} .$$

This shows $\| \chi_A (f_n - f) \|_{L^q} \to 0$ as $n \to \infty$ if $1 \le q \le 2$.
Assume now $2 < q < \frac{2d}{d-2}$ if $d \ge 3$, or $2 \le q < \infty$ if $d = 1, 2$. We then need a different interpolation, since we would have $\frac{1}{\alpha'} < 0$. One can show the validity of the interpolation inequality

$$\| \chi_A (f_n - f) \|_{L^q} \le \| \chi_A (f_n - f) \|_{L^2}^{\theta} \, \| \chi_A (f_n - f) \|_{L^{q_0}}^{1-\theta}$$

with $q_0 = \frac{2d}{d-2}$ and $\theta \in (0,1)$ is a suitable number. This is the right inequality for $d \ge 3$: the L^2-norm converges to zero, while for the L^{q_0}-norm we use the Sobolev embedding

$$\| \chi_A (f_n - f) \|_{L^{q_0}} \le \| f_n - f \|_{L^{q_0}} \le \| f_n \|_{L^{q_0}} + \| f \|_{L^{q_0}}$$

$$\le C \left(\| \nabla f_n \|_{L^{q_0}} + \| \nabla f \|_{L^{q_0}} \right) \le C .$$

This proves $\| \chi_A (f_n - f) \|_{L^q} \to 0$ as $n \to \infty$ if $d \ge 3$ and $2 \le q < \frac{2d}{d-2}$.
For $d = 1, 2$ one argues similarly again with the inequality

$$\| \chi_A (f_n - f) \|_{L^q} \le \| \chi_A (f_n - f) \|_{L^2}^{\theta} \, \| \chi_A (f_n - f) \|_{L^{q_0}}^{1-\theta} ,$$

where now $q_0 > q$ is any number, and $\theta \in (0,1)$ is accordingly modified. As above, the L^2-norm converges to zero, while the L^{q_0}-norm is controlled by $\| \nabla f_n \|_{L^2} + \| \nabla f \|_{L^2}$ thanks to the Sobolev embedding.

(5) Uniform convergence in $d = 1$.

If $d = 1$, the fact that $f_n, f \in H^1(\mathbb{R}^d)$ implies, by Theorem 3.12, that f_n, f are Hölder continuous with exponent $1/2$.

Moreover, since $f_n' \in L^2(\mathbb{R}^d)$ it is in particular locally integrable. We can thus apply the fundamental theorem of calculus. We have

$$f_n(x) - f_n(0) = \int_0^x f_n'(s)\,ds \longrightarrow \int_0^x f'(s)\,ds = f(x) - f(0).$$

since $f_n' \rightharpoonup f'$ weakly in $L^2(\mathbb{R}^d)$. Recalling that the Sobolev embedding in $d=1$ allows even for $q=\infty$, i.e.

$$\| h \|_{L^\infty(\mathbb{R})} \leq C \| h \|_{H^1(\mathbb{R})} ,$$

we find $|f_n(x)| \leq C \| f_n \|_{H^1} \leq C \quad \forall x \in \mathbb{R}$ uniformly in n, since $f_n \rightharpoonup f$ weakly in H^1 and therefore $\|f_n\|_{H^1} \leq C$.
We thus have, $\forall g \in L^1(\mathbb{R})$,

$$\lim_{n\to\infty} \int_{\mathbb{R}} g(x)\big(f_n(x) - f_n(0)\big)dx = \int_{\mathbb{R}} g(x)\big(f(x) - f(0)\big)dx$$

by an immediate application of dominated convergence. If, moreover, $g \in L^2(\mathbb{R})$, we also have

$$\lim_{n\to\infty} \int_{\mathbb{R}} g(x) f_n(x)\,dx = \int g(x) f(x)\,dx$$

by the weak convergence $f_n \rightharpoonup f$ in $L^2(\mathbb{R})$. Choosing then $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, we find by the two above identities

$$f_n(0) \to f(0), \text{ and thus } f_n(x) \to f(x) \text{ pointwise for } x \in \mathbb{R}.$$

We now have to show that convergence is uniform on compact sets. Assume by contradiction that I is a compact interval where f_n does not converge uniformly. Then $\exists\, \varepsilon > 0$ and a sequence $x_n \in I$ with

$$|f_n(x_n) - f(x_n)| > \varepsilon \qquad \forall n \in \mathbb{N}.$$

Since I is compact, we can assume, up to a subsequence, that $x_n \to x$ for some $x \in I$. Recall now that, by Theorem 3.12,

$$|f_n(x) - f_n(y)| \leq \| f_n' \|_{L^2} |x-y|^{\frac{1}{2}} \leq C\,|x-y|^{\frac{1}{2}} \quad \forall x,y \in \mathbb{R},$$

where the last step follows once again by boundedness, since $f_n' \to f'$ weakly in L^2. Clearly we also have

$$|f(x) - f(y)| \leq C\,|x-y|^{\frac{1}{2}} \qquad \forall x,y \in \mathbb{R}.$$

But then, $\forall n \in \mathbb{N}$,

$$\begin{aligned} \varepsilon &< |f_n(x_n) - f(x_n)| \\ &\leq |f_n(x_n) - f_n(x)| + |f_n(x) - f(x)| + |f(x) - f(x_n)| \\ &\leq C|x-x_n|^{\frac{1}{2}} + |f_n(x) - f(x)|, \end{aligned}$$

contradicting $x_n \to x$ or $f_n \to f$ pointwise. ∎

The last topic of this section is the general statement for compact Sobolev embeddings. This is known as the Rellich-Kondrachov Theorem. To put it into context, it is worth recalling the main content of Theorem 3.14, i.e., the general Sobolev inequalities.

The main players are:

- $\Omega \subseteq \mathbb{R}^d$ open and with the cone property.
- p,q with $1 \leq p \leq q \leq \infty$. L^p will play the role of the "low p space" as was L^2 for the inequalities for H^1. L^q will play the role of "high q space" as was $2 \leq q \leq \frac{2d}{d-2}$ if $d \geq 3$ for the H^1 case.
- $m, k \in \mathbb{N}$ with $k \leq m$, $m \geq 1$. m will be the number of derivatives in the right, think of $m=1$ for H^1. k will be the difference in the number of derivatives between right and left, i.e., the derivatives that one can "pay" to control $W^{m-k,q}$ with $W^{m,p}$.

There was a difference in the p,q allowed, depending on k,d.

1) If $kp < d$, then $\quad \|f\|_{W^{m-k,q}(\Omega)} \leq C\|f\|_{W^{m,p}(\Omega)}$

for all $p \leq q \leq \dfrac{dp}{d-kp}$. This is the generalization of $\|\cdot\|_{L^q} \leq \|\cdot\|_{H^1}$ in $d \geq 3$ and $2 \leq q \leq \frac{2d}{d-2}$.

2) If $kp=d$ (critical case), then $\|f\|_{W^{m-k,q}(\Omega)} \leq C\|f\|_{W^{m,p}(\Omega)}$ for all $p \leq q < \infty$.
Think of $d=2$ and $\|\cdot\|_{L^q} \leq \|\cdot\|_{H^1}$

3) If $kp>d$, then $\|f\|_{W^{m-k,q}(\Omega)} \leq C\|f\|_{W^{m,p}(\Omega)}$ for all $p \leq q \leq \infty$.

The following theorem shows that, away from the critical exponent, and restricting to a bounded set, weakly convergent sequences in $W^{m,p}$ converge strongly in $W^{m-k,p}$.

Theorem 4.15 (Rellich-Kondrachov)

Let $\Omega \subset \mathbb{R}^d$ be open and with the cone property. Let $1 \leq p \leq q \leq \infty$, $m, k \in \mathbb{N}$ with $m \geq 1$ and $k \leq m$. Let $f_n, f \in W^{m,p}(\Omega)$ with $f_n \to f$ weakly in $W^{m,p}(\Omega)$ (which means $D^\alpha f_n \to D^\alpha f$ weakly in $L^p(\Omega)$ for all multiindices $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq m$). Let $\omega \subset \Omega$ be open and bounded. Then

1) If $kp<d$ and $1 \leq q < \frac{dp}{d-kp}$, then $\|f_n - f\|_{W^{m-k,q}(\omega)} \to 0$ as $n \to \infty$.

2) If $kp=d$ and $1 \leq q < \infty$, then $\|f_n - f\|_{W^{m-k,q}(\omega)} \to 0$ as $n \to \infty$.

3) If $kp>d$ and $1 \leq q \leq \infty$, then $\|f_n - f\|_{W^{m-k,q}(\omega)} \to 0$ as $n \to \infty$.

As already discussed early in the section, the above theorem is sometimes stated by saying that the unit ball of $W^{m,p}(\Omega)$ is a compact subset of $W^{m-k,q}(\Omega)$ in the three appropriate regime of indices. In other words, every bounded sequence in $W^{m,p}(\Omega)$ has a subsequence that converge strongly in $W^{m-k,q}(\Omega)$.