(3) Sobolev Spaces

Sobolev spaces are very important spaces of functions, with natural applications in PDE, mathematical physics, and other fields. Here is a possible motivating example.
The time-independent Schrödinger equation for a quantum particle on $[0,1]$ is

$$-\Delta \psi = E\psi$$

Here $\psi \in L^2([0,1])$ is the unknown function, and E is its energy. If $\|\psi\|_{L^2} = 1$, then quantum mechanics interprets the measure $|\psi^2(x)|\,dx$ as the probability density of finding a quantum particle at x.
Such equation is closely linked to the minimization problem

$$\inf\left\{ \int_0^1 |\nabla\psi(x)|^2 dx \;\middle|\; \psi \in L^2([0,1]) \text{ such that } \|\psi\|_{L^2} = 1 \text{ and such that } \int_0^1 |\nabla\psi|^2 < +\infty \right\}.$$

In fact, such a number (the infimum) coincides with the minimal E for which a solution exists. We will study all this in greater detail later. For the moment however: what are the ψ's for which $\int_0^1 |\nabla\psi|^2$ exists and is finite? These are precisely functions in $W^{1,2}([0,1])$: functions in L^2 with derivative in L^2.
This was just one among a myriad examples of the fact that many PDE are naturally posed in some Sobolev space.

<u>Definition 3.1 (Sobolev spaces $W^{k,p}(\Omega)$)</u>
Let $k \in \mathbb{N}$ and $1 \le p \le +\infty$. Let $\Omega \subseteq \mathbb{R}^d$ be an open set. The space $W^{k,p}(\Omega)$ is the space of functions $f \in L^p(\Omega)$ such that $\partial^\alpha f \in L^p(\Omega)$ for every multiindex $|\alpha| \le k$. $W^{k,p}(\Omega)$ is a normed space when endowed with the norm

$$\|f\|_{W^{k,p}(\Omega)} := \left(\sum_{|\alpha| \le k} \|\partial^\alpha f\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} \quad \text{for } p < +\infty$$

and

$$\|f\|_{W^{k,\infty}(\Omega)} := \sup_{|\alpha|\leq k} \|\partial^\alpha f\|_{L^\infty(\Omega)}.$$

The case $p=2$ has special properties.

Definition 3.1 ($H^k(\Omega)$)

We define $H^k(\Omega) := W^{k,2}(\Omega)$, with norm $\|\cdot\|_{H^k} := \|\cdot\|_{W^{k,2}}$.
Its norm is induced by the scalar product (sesquilinear form)

$$\langle f, g \rangle_{H^k(\Omega)} := \sum_{|\alpha|\leq k} \int_\Omega \overline{\partial^\alpha f(x)}\, \partial^\alpha g(x)\, dx.$$

The space $H^1(\mathbb{R}^d)$ will be the most important all through the course.

One of the crucial properties of Sobolev spaces is the fact that they are complete (i.e. Banach spaces).

Theorem 3.3 (Completeness of $W^{k,p}(\Omega)$)

The spaces $W^{k,p}(\Omega)$ are complete, i.e., for any sequence $f_n \in W^{k,p}(\Omega)$ such that

$$\|f_n - f_m\|_{W^{k,p}(\Omega)} \to 0 \quad \text{as } n, m \to \infty,$$

there exists $f \in W^{k,p}(\Omega)$ such that

$$\lim_{n\to\infty} \|f_n - f\|_{W^{k,p}(\Omega)} = 0.$$

Proof

For simplicity we will prove this for $H^1(\Omega)$ only.
Let f_n be a Cauchy sequence in $H^1(\Omega)$. By definition of H^1 and its norm, f_n is a Cauchy sequence in $L^2(\Omega)$ as well. Since $L^2(\Omega)$ is a Banach space, f_n must have a limit in $L^2(\Omega)$, i.e., there exists $f \in L^2(\Omega)$ such that

$$\lim_{n\to\infty} \|f-f_n\|_{L^2} = 0.$$

We however want to show convergence of derivatives as well. Again by definition of $H^1(\Omega)$, $\nabla f_n \in L^2(\Omega)$ and it is a Cauchy sequence by assumption. It must thus have a limit in $L^2(\Omega)$

$$\lim_{n\to\infty} \nabla f_n = (b_1, \dots, b_d) = b \text{ in } L^2(\Omega), \text{ with } \Omega \subseteq \mathbb{R}^d.$$

The proof is complete if we are able to show that $b = \nabla f$ in $\mathcal{D}'(\Omega)$. Let us thus fix $\phi \in \mathcal{D}(\Omega)$. By Cauchy-Schwartz (or Hölder) inequality,

$$\left| \int_\Omega \nabla\phi(x)(f(x) - f_n(x))\,dx \right| \leq \|\nabla\phi\|_{L^2} \|f-f_n\|_{L^2} \leq C\,\|f-f_n\|_{L^2}$$

we deduce

$$\lim_{n\to\infty} \int_\Omega \nabla\phi(x)\, f_n(x)\,dx = \int_\Omega \nabla\phi(x)\, f(x)\,dx.$$

In the same way we can prove that

$$\int_\Omega \phi(x)\, b(x)\,dx = \lim_{n\to\infty} \int_\Omega \phi(x)\, \nabla f_n(x)\,dx.$$

We thus deduce

$$-T_{\nabla f}(\phi) = \int_\Omega \nabla\phi(x)\, f(x)\,dx = \lim_{n\to\infty} \int_\Omega \nabla\phi(x)\, f_n(x)\,dx$$

$$= -\lim_{n\to\infty} \int_\Omega \phi(x)\, \nabla f_n(x)\,dx = -\int_\Omega \phi(x)\, b(x)\,dx = -T_b(\phi).$$

This is precisely the property $\nabla f = b$ in $\mathcal{D}'(\Omega)$, which concludes the proof. Notice that the first and third equalities above follow by definition of weak derivative, since $f, f_n, \nabla f_n \in L^1_{loc}(\Omega)$. ∎

We aim now at showing that smooth functions are dense in Sobolev spaces. Recall that this is true for L^p spaces. In principle one might wonder why we need to show it again for a subset of L^p spaces. The reason is the fact that we will prove density in Sobolev norm, a much stronger result.

Theorem 3.4 (Density of $C^\infty(\Omega)$ in $H^1(\Omega)$)

For every $f \in H^1(\Omega)$, there exists a sequence $f_n \in C^\infty(\Omega) \cap H^1(\Omega)$ such that

$$\lim_{n\to\infty} \|f_n - f\|_{H^1(\Omega)} = 0.$$

If $\Omega = \mathbb{R}^d$, then f_n can be taken in $C_c^\infty(\Omega)$ for all n.

It is worth remarking that the above result holds for all Sobolev spaces $W^{k,p}(\Omega)$ with $p<+\infty$ (Meyers-Serrin Theorem).
Notice also that, in some texts, what we called $H^1(\Omega)$ and indistinguishably $W^{1,2}(\Omega)$, is only called $W^{1,2}(\Omega)$. The notation $H^1(\Omega)$ is used for the completion of $C^\infty(\Omega)$ under the Sobolev norm of $W^{1,2}(\Omega)$.
These two definitions are equivalent, as shown by Theorem 3.4.
We will prove Theorem 3.4 for the case $\Omega = \mathbb{R}^d$ only. We will need the next lemma, whose proof is left as exercise.

Lemma 3.5 (Multiplication by $C^\infty(\Omega)$ functions)

Let $f \in H^1(\Omega)$ and $\psi \in C^\infty(\Omega)$ bounded with bounded derivatives. Then the pointwise product $(\psi f)(x) = \psi(x) f(x)$ belongs to $H^1(\Omega)$ and

$$\frac{\partial}{\partial x_i}(\psi f) = \frac{\partial \psi}{\partial x_i} f + \psi \frac{\partial f}{\partial x_i} \quad \text{in } \mathcal{D}'(\Omega).$$

Proof of Theorem 3.4

The first goal is to prove density of $C^\infty(\mathbb{R}^d)$. This is achieved by a mollification argument, as presented at the beginning of the course.
Given $J : \mathbb{R}^d \to \mathbb{R}^+$ in $C_c^\infty(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} J = 1$, and defining $J_\varepsilon(x) = \varepsilon^{-d} J(\frac{x}{\varepsilon})$, we know from Theorem 0.6 that

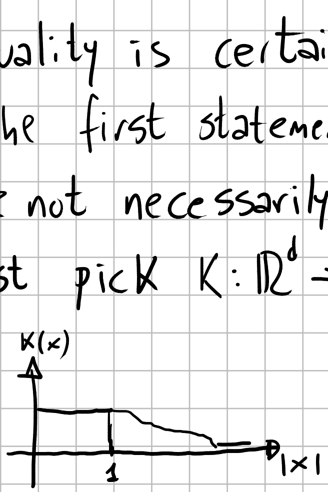
$$f_\varepsilon := J_\varepsilon * f \to f \ \text{ in } L^2(\mathbb{R}^d) \quad \text{and} \quad g_\varepsilon := J_\varepsilon * \nabla f \to \nabla f \ \text{ in } L^2(\mathbb{R}^d).$$

If $g_\varepsilon = \nabla(f_\varepsilon)$, then we would have $f_\varepsilon \to f$ and $\nabla(f_\varepsilon) \to \nabla f$ in $L^2(\mathbb{R}^d)$, which means $f_\varepsilon \to f$ in $H^1(\mathbb{R}^d)$. The fact that

$$g_\varepsilon = J_\varepsilon * (\nabla f) = \nabla (J_\varepsilon * f) = \nabla f_\varepsilon$$

is a standard property of distributions and weak derivatives, whose proof is left as exercise (see Lemma 6.8 in Lieb-Loss). Notice that the above equality is certainly true for regular functions.

The first statement is proven by choosing, for example, $\varepsilon = \frac{1}{m}$. The f_ε are not necessarily compactly supported though. To achieve this, let us first pick $K : \mathbb{R}^d \to [0,1]$ in $C_c^\infty(\mathbb{R}^d)$ with $K(x) = 1$ for $|x| \leq 1$,



and define $h_n(x) := K(\frac{x}{n}) f(x)$.

This is a compactly supported approximation of f such that $h_n \in H^1(\mathbb{R}^d)$ $\forall n$ by Lemma 3.5 and

$$\| f - h_n(x) \|_{L^2}^2 \leq C \int_{|x| \geq n} |f(x)|^2 \, dx \to 0 \quad \text{as } n \to \infty$$

$$\| \nabla f - \nabla h_n \|_{L^2}^2 = \int_{\mathbb{R}^d} | \nabla f(x) - \nabla f(x) K(\tfrac{x}{n}) - \nabla K(\tfrac{x}{n}) f(x) |^2 dx$$

$$\leq C \int_{|x| \geq n} |\nabla f(x)|^2 dx + \frac{C}{n^2} \int_{\mathbb{R}^d} |f(x)|^2 dx \to 0 \quad n \to \infty.$$

Here we used $|\nabla K(\frac{x}{n})| = |\frac{1}{n} (\nabla K)(\frac{x}{n})| \leq \frac{C}{n}$. This shows $h_n \to f$ in $H^1(\mathbb{R}^d)$ as $n \to \infty$. We now define the sequence in $C_c^\infty(\mathbb{R}^d)$ that approximates f as

$$F_n(x) = K(\tfrac{x}{n}) \cdot f_{1/n}(x).$$

Then $$\| f - F_n \|_{H^1} = \| f - K(\tfrac{\cdot}{n}) \cdot f_{1/n} \|_{H^1} \leq \| f - h_n \|_{H^1} + \| h_n - K(\tfrac{\cdot}{n}) f_{1/n} \|_{H^1}$$

$$\leq \| f - h_n \|_{H^1} + \| K(\tfrac{\cdot}{n})(f - f_{1/n}) \|_{L^2} + \| \nabla (K(\tfrac{\cdot}{n})(f - f_{1/n})) \|_{L^2}.$$

The first summand converges to zero by what we have shown above. The second one converges to zero as well because K is bounded and $f_{\frac{1}{n}} \to f$ in $L^2(\mathbb{R}^d)$. For the third one we write (using Lemma 3.5)

$$\| \nabla (K(\tfrac{\cdot}{n})(f - f_{1/n})) \|_{L^2}^2 = \int_{\mathbb{R}^d} | (\nabla K(\tfrac{x}{n}))(f(x) - f_{\frac{1}{n}}(x)) + K(\tfrac{x}{n}) \nabla (f(x) - f_{\frac{1}{n}}(x)) |^2 dx$$

$$\leq 2 \int_{\mathbb{R}^d} |\nabla K(\tfrac{x}{n})|^2 \, |f(x) - f_{1/n}(x)|^2 dx$$

$$+2\int_{\mathbb{R}^d} |K(\tfrac{x}{n})|^2 \cdot |\nabla(f(x) - f_{1/n}(x))|^2 dx.$$

$$\leq \frac{C}{n^2}\int_{\mathbb{R}^d} |f - f_{1/n}|^2 + \int_{\mathbb{R}^d} |\nabla f - \nabla f_{1/n}|^2.$$

Both summands tend to zero since $f_{1/n} \to f$ in $H^1(\mathbb{R}^d)$. ■

We continue the characterization of functions in $H^1(\mathbb{R}^d)$ and their properties with the next result.

Theorem 3.6 (Integration by parts)

Let $u, v \in H^1(\mathbb{R}^d)$. Then, for any $j = 1, \dots, d$, we have

$$\int_{\mathbb{R}^d} u(x) \frac{\partial v}{\partial x_j}(x)\, dx = -\int_{\mathbb{R}^d} \frac{\partial u}{\partial x_j}(x)\, v(x) dx.$$

Suppose, in addition, that $v \in W^{2,2}(\mathbb{R}^d) = H^2(\mathbb{R}^d)$. Then

$$-\int_{\mathbb{R}^d} u(x)\, \Delta v(x)\, dx = \int_{\mathbb{R}^d} \nabla u(x) \cdot \nabla v(x) dx.$$

It is worth remarking that the assumption $v \in H^2(\mathbb{R}^d)$ can be relaxed a little, by allowing for Δv to have a $L^1_{loc}(\mathbb{R}^d)$ part.

The above statement plays a role in the study of Schrödinger equations of the type

$$-\Delta \psi + V\psi = E\psi \quad \text{in } \mathcal{D}'(\mathbb{R}^d).$$

We might want to rewrite this, for some $\varphi \in H^1(\mathbb{R}^d)$ (not $\mathcal{D}(\mathbb{R}^d)$!), as

$$\int_{\mathbb{R}^d} \nabla\varphi \cdot \nabla\psi + \int_{\mathbb{R}^d} V\varphi\psi = E\int_{\mathbb{R}^d} \varphi\psi.$$

This will be allowed (for suitable V) by Theorem 3.5.

Proof of Theorem 3.6

Notice that all 4 functions in the first statement are L^2, and therefore the integrals make at least sense. By Theorem 3.4 $\exists\, u_n \in C_c^\infty(\mathbb{R}^d)$ such that $\|u_n - u\|_{H^1} \to 0$ as $n \to \infty$. We also know that

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} u_n(x) \frac{\partial v}{\partial x_j}(x)\,dx = \int_{\mathbb{R}^d} u(x) \frac{\partial v}{\partial x_j}(x)\,dx$$

since, by Cauchy-Schwarz's inequality,

$$\left| \int_{\mathbb{R}^d} (u_n(x) - u(x)) \frac{\partial v}{\partial x_j}(x)\,dx \right| \leq \| u - u_n \|_{L^2} \left\| \frac{\partial v}{\partial x_j} \right\|_{L^2} \to 0 \quad \text{as } n \to \infty.$$

And, again by the same argument,

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \frac{\partial u_n}{\partial x_j}(x)\, v(x)\,dx = \int \frac{\partial u}{\partial x_j}(x)\, v(x)\,dx.$$

We can thus write

$$\int_{\mathbb{R}^d} u(x) \frac{\partial v}{\partial x_j}(x) = \lim_{n\to\infty} \int u_n(x) \frac{\partial v}{\partial x_j}(x)\,dx = \lim_{n\to\infty} T_{\frac{\partial v}{\partial x_j}}(u_n)$$

$$= -\lim_{n\to\infty} T_v(\partial_j u_n) = -\lim_{n\to\infty} \int_{\mathbb{R}^d} \frac{\partial u_n}{\partial x_j}(x)\, v(x)\,dx$$

$$= -\int_{\mathbb{R}^d} \frac{\partial u}{\partial x_j}(x)\, v(x)\,dx.$$

This proves the first claim.

To prove that $-\int u \Delta v = \int \nabla u \cdot \nabla v$, we use density of $C_c^\infty(\mathbb{R}^d)$ functions in $H^1(\mathbb{R}^d)$ and in $W^{2,2}(\mathbb{R}^d) = H^2(\mathbb{R}^d)$ (we didn't prove this, but claimed that it holds). It means that $\exists\, u_n, v_n \in C_c^\infty(\mathbb{R}^d)$ such that

$$u_n \xrightarrow[n\to\infty]{} u \text{ in } H^1(\mathbb{R}^d), \qquad v_n \xrightarrow[n\to\infty]{} v \text{ in } H^2(\mathbb{R}^d).$$

In particular, $\|\Delta v_n - \Delta v\|_{L^2} \to 0$. We can thus claim that, since the statement certainly holds for C_c^∞-functions, it can be extended to u, v as in the statement through a density argument as above $\blacksquare$

We also present the following further property of functions in $H^1(\mathbb{R}^d)$.

Theorem 3.7 (Convexity inequality for gradients)

Let f, g be real-valued and in $H^1(\mathbb{R}^d)$. Then

$$\int_{\mathbb{R}^d} \left|\nabla \sqrt{f^2+g^2}\right|^2(x)\,dx \leq \int_{\mathbb{R}^d} \left(|\nabla f|^2(x) + |\nabla g|^2(x)\right)dx.$$

If $g>0$ a.e., then equality holds if and only if $\exists c$ constant such that $f(x)=c\,g(x)$ a.e.

It is important to remark that the statement is equivalent to

$$\int_{\mathbb{R}^d} |\nabla |F(x)||^2\,dx \leq \int_{\mathbb{R}^d} |\nabla F(x)|^2\,dx$$

for a complex-valued $H^1(\mathbb{R}^d)$-function F.

Indeed, if the statement holds, then we simply apply it for $f = \mathrm{Re}F$, $g = \mathrm{Im} f$. If, instead, the property above holds for every F, then we apply it for $F = f + ig$.

The take-home message of Theorem 3.7 is that passing from F to $|F|$ decreases the H^1 norm.

In the proof of Theorem 3.7 we will use the following result, whose proof will be discussed in the exercise classes.

Lemma 3.8 (Derivative of the absolute value)

Let $f \in W^{1,p}(\Omega)$ for $\Omega \subseteq \mathbb{R}^d$ open. Then $|f| \in W^{1,p}(\Omega)$, with $|f|(x) = |f(x)|$ and

$$(\nabla |f|)(x) = \begin{cases} \dfrac{1}{|f(x)|}\Big((\mathrm{Re} f)(x)\,\nabla(\mathrm{Re} f)(x) + (\mathrm{Im} f)(x)\,\nabla(\mathrm{Im} f)(x)\Big) & f(x) \neq 0 \\ 0 & f(x) = 0. \end{cases}$$

Notice that this implies $|\nabla |f|| = |\nabla f|$ if f is real-valued, and $|\nabla |f|| \leq |\nabla f|$ if f is generic complex-valued. Indeed,

$$|\nabla |f|| = \frac{1}{|f|} \left| \mathrm{Re} f \, \nabla(\mathrm{Re} f) + \mathrm{Im} f \, \nabla(\mathrm{Im} f)\right|$$

$$= \frac{1}{|f|} \left[(\mathrm{Re} f)^2 (\nabla \mathrm{Re} f)^2 + (\mathrm{Im} f)^2 (\nabla \mathrm{Im} f)^2 + 2 (\mathrm{Re} f)(\mathrm{Im} f)(\nabla \mathrm{Re} f)(\nabla \mathrm{Im} f)\right]^{1/2}$$

$$\leq \frac{1}{|f|} \Big[(\mathrm{Re} f)^2 (\nabla \mathrm{Re} f)^2 + (\mathrm{Im} f)^2 (\nabla \mathrm{Im} f)^2$$
$$+ (\mathrm{Re} f)^2 (\nabla \mathrm{Im} f)^2 + (\mathrm{Im} f)^2 (\nabla \mathrm{Re} f)^2 \Big]^{1/2}$$

$$= (\nabla \mathrm{Re} f)^2 + (\nabla \mathrm{Im} f)^2 = |\nabla f|.$$

You might encounter a slight generalization of Lemma 3.7, sometimes called diamagnetic inequality:

$$|\nabla |F|| \leq |(\nabla + iA) F|$$

valid for $F \in H^1(\mathbb{R}^d)$ and for $A : \mathbb{R}^d \to \mathbb{R}^d$ in $L^2_{loc}(\mathbb{R}^d)$.

Notice also that the assumption $g > 0$ is not necessarily obvious for a function defined almost everywhere. We mean here, by definition, that for every compact $K \subset \mathbb{R}^d$, $\exists\, \varepsilon > 0$ such that $\{x \in K \text{ s.t. } g(x) < \varepsilon\}$ has measure zero.

This forbids situations of the type $f(x) = \begin{cases} x^2 & x \neq 0 \\ 1 & x = 0 \end{cases}$. This function is clearly "positive" but not in the above sense.

Proof of Theorem 3.7

The function $\sqrt{f^2+g^2}$ coincides with $|f+ig|$. Since $f+ig \in H^1(\mathbb{R}^d)$, by Lemma 3.6 we have $\sqrt{f^2+g^2} \in H^1(\mathbb{R}^d)$ and

$$\left(\nabla\sqrt{f^2+g^2}\right)(x) = \begin{cases} \dfrac{f(x)\nabla f(x) + g(x)\nabla g(x)}{\sqrt{f^2(x)+g^2(x)}} & f^2(x)+g^2(x) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

We now claim the followin identity:

$$\int_{\mathbb{R}^d} \left|\nabla\sqrt{f^2+g^2}\right|^2 + \int_{f^2+g^2>0} \frac{|g\nabla f - f\nabla g|^2}{f^2+g^2} = \int_{\mathbb{R}^d} \left(|\nabla f|^2 + |\nabla g|^2\right).$$

If this is true, then the first statement immediately follows since the right hand side is larger than $\int \left|\nabla\sqrt{f^2+g^2}\right|^2$. To prove the above identity we directly check

$$\int_{\mathbb{R}^d} \left|\nabla\sqrt{f^2+g^2}\right|^2 + \int_{f^2+g^2>0} \frac{|g\nabla f - f\nabla g|^2}{f^2+g^2}$$

$$= \int_{f^2+g^2>0} \frac{1}{f^2+g^2}\left[f^2|\nabla f|^2 + g^2|\nabla g|^2 + \cancel{2fg\nabla f\cdot\nabla g} + g^2|\nabla f|^2 + f^2|\nabla g|^2 - \cancel{2fg\nabla f\cdot\nabla g}\right]$$

$$= \int_{f^2+g^2>0} \left(|\nabla f|^2 + |\nabla g|^2\right) = \int_{\mathbb{R}^d} \left(|\nabla f|^2 + |\nabla g|^2\right).$$

The last inequality follows from the fact that $\nabla w(x)=0$ for almost every $x \in w^{-1}(A)$, where A is a set of measure zero (e.g. $A=\{0\}$), and $w \in W^{1,1}_{loc}(\Omega)$ for $\Omega \subseteq \mathbb{R}^d$ open (Theorem 6.19 Lieb-Loss).
This is quite clear if $w^{-1}(A)$ is open. But it is true even if $w^{-1}(A)$ is uglier (e.g. Cantor set,..).
We now have to prove the last statement in Theorem 3.7. Assume thus that $g>0$, and that

$$\int |\nabla \sqrt{f^2+g^2}|^2 = \int (|\nabla f|^2 + |\nabla g|^2)$$

By the above identity again, this means $g\nabla f = f \nabla g$ almost everywhere in $\mathbb{R}^d$. Let $\phi \in C^\infty_c(\mathbb{R}^d)$, and consider $h(x) := \phi(x)/g(x)$.
Let us first of all check that $h \in L^2(\mathbb{R}^d)$:

$$\|h\|^2_{L^2} = \int_{\mathbb{R}^d} \left| \frac{\phi(x)}{g(x)} \right|^2 \leq C \int_{\operatorname{supp}\phi} \frac{1}{|g(x)|^2} \leq \frac{C}{\varepsilon^2} |\operatorname{supp}\phi| ,$$

where, by definition of $g>0$, $\exists\, \varepsilon>0$ such that $g>\varepsilon$ a.e. on $\operatorname{supp}\phi$.
It turns out that actually $h \in H^1(\mathbb{R}^d)$. To see this, define the approximation of h

$$h_\delta(x) = \frac{\phi(x)}{\sqrt{g^2(x)+\delta^2}} .$$

This is manifestly in $H^1(\mathbb{R}^d)$, since we can apply the chain rule (Theorem 6.16 in Lieb-Loss, valid under minimal requirements, for which we add the δ) to write

$$\nabla h_\delta(x) = -\frac{\phi(x)\nabla g(x)}{g^2(x)+\delta^2} + \frac{\nabla\phi(x)}{\sqrt{g^2(x)+\delta^2}} .$$

Let now $\psi \in \mathcal{D}(\mathbb{R}^d)$. Then

$$T_{\nabla h}(\psi) = -T_h(\nabla\psi) = -\int_{\mathbb{R}^d} h(x)\nabla\psi(x)\,dx = -\lim_{\delta\to 0}\int_{\mathbb{R}^d} h_\delta(x)\nabla\psi(x)\,dx .$$

Passing to the limit is allowed since

$$\|h-h_\delta\|^2_{L^2} = \int_{\operatorname{supp}\phi} |\phi(x)|^2 \left[\frac{1}{g(x)} - \frac{1}{\sqrt{g(x)^2+\delta^2}}\right]^2 dx .$$

Indeed, by assumption, $\exists \varepsilon$ s.t. $g > \varepsilon$ a.e. on $\operatorname{supp}\phi$, we have that the integrand satisfies

$$|\phi(x)|^2 \left[\frac{1}{g(x)} - \frac{1}{\sqrt{g(x)^2+\delta^2}} \right]^2 \leq 2|\phi(x)|^2 \left(\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon^2+\delta^2} \right)$$

$$\leq \frac{4}{\varepsilon^2} |\phi(x)|^2 \quad \text{a.e. on } \operatorname{supp}\phi.$$

Since the integrand also converges pointwise a.e. to zero as $\delta \to 0$, we have $\|h - h_\delta\|_{L^2} \to 0$ by dominated convergence.

We can now integrate by parts in $\int h_\delta \nabla\psi$ and use the expression above for ∇h_δ to get

$$T_{\nabla h}(\varphi) = \lim_{\delta \to 0} \int_{\mathbb{R}^d} \left(\frac{\nabla\phi(x)}{\sqrt{g^2(x)+\delta^2}} - \frac{\phi(x)\nabla g(x)}{g^2(x)+\delta^2} \right) \psi(x)\, dx.$$

A similar dominated convergence argument shows that we can again pass to the limit. This proves

$$\nabla h = -\frac{\nabla g}{g^2}\phi + \frac{\nabla\phi}{g} \quad \text{in } \mathcal{D}'(\mathbb{R}^d).$$

Since the right hand side is the sum of two L^2 functions (again because $g > 0$), the two sides must coincide in L^2 as well. This proves $h \in H^1(\mathbb{R}^d)$. We now write

$$\int_{\mathbb{R}^d} f(x) \nabla h(x)\, dx = -\int_{\mathbb{R}^d} f(x) \frac{\nabla g(x)}{g^2(x)} \phi(x) dx + \int_{\mathbb{R}^d} f(x) \frac{\nabla\phi(x)}{g(x)} dx$$

$$= -\int_{\mathbb{R}^d} \nabla f(x)\, h(x) dx + \int_{\mathbb{R}^d} \frac{f(x)}{g(x)} \nabla\phi(x) dx,$$

having used $f \nabla g = g \nabla f$ a.e., which was proven above. By Theorem 3.6 we can integrate by parts:

$$\int_{\mathbb{R}^d} \nabla f(x)\, h(x)\, dx = -\int_{\mathbb{R}^d} f(x) \nabla h(x) dx.$$

This implies $\displaystyle\int_{\mathbb{R}^d} \frac{f(x)}{g(x)} D\phi(x)\,dx = 0$.

Now, $g>0$ implies that $\frac{f}{g} \in L^1_{loc}(\mathbb{R}^d)$ and thus $\frac{f}{g}$ is a distribution. Since ϕ is a test function, we have proven

$$D\left(\frac{f}{g}\right)=0 \quad \text{in } \mathcal{D}'(\mathbb{R}^d).$$

But distributions with zero derivative are constant functions a.e. (Theorem 2.8) ■

Anothe very important result is the following characterization of $H^1(\mathbb{R}^d)$. It shows that $H^1(\mathbb{R}^d)$ is actually isomorphic to an L^2 space, but with the measure $(1+x^2)dx$ replacing the Lebesgue measure. The isomorphism is provided by the Fourier transform.

Theorem 3.9 (Fourier characterization of $H^1(\mathbb{R}^d)$)

Let $f \in L^2(\mathbb{R}^d)$ and $\hat{f} \in L^2(\mathbb{R}^d)$ be its Fourier transform. Then $f \in H^1(\mathbb{R}^d)$ if and only if the function $k \longmapsto |k|\,\hat{f}(k)$ is in $L^2(\mathbb{R}^d)$. In this case we have

$$\widehat{\nabla f}(k) = 2\pi i k \hat{f}(k)$$

and therefore

$$\|f\|^2_{H^1} = \int_{\mathbb{R}^d} |\hat{f}(k)|^2 (1+4\pi^2|k|^2)\,dk$$

The above result could be extended to all $H^s(\mathbb{R}^d)$'s. In fact,

$$H^s(\mathbb{R}^d) = \{ f\in L^2(\mathbb{R}^d) \mid k \longmapsto |k|^s \hat{f}(k) \text{ belongs to } L^2(\mathbb{R}^d) \}$$

for any $s \in \mathbb{N}$. This has an immediate extension to every $s>0$.

Proof

Suppose $f \in H^1(\mathbb{R}^d)$. Let us prove that $k \longmapsto |k|\hat{f}(k)$ belongs to

$L^2(\mathbb{R}^d)$. By Theorem 3.4 there exists $f_n \in C_c^\infty(\mathbb{R}^d)$ such that $f_n \to f$ in $H^1(\mathbb{R}^d)$. Since $f_n \in C_c^\infty(\mathbb{R}^d)$, it belongs in particular to $S(\mathbb{R}^d)$, and therefore by Proposition 2.11 (or check directly integrating by parts)

$$\widehat{\nabla f_n}(K) = 2\pi i K\, \hat{f}_n(K).$$

Now, since $\|f_n - f\|_{H^1} \to 0$, we have, by Plancherel's Theorem,

$$\|\hat{f}_m - \hat{f}\|_{L^2} = \|f_n - f\|_{L^2} \to 0$$

and

$$\|\widehat{\nabla f}_m - \widehat{\nabla f}\|_{L^2} = \|\nabla f_m - \nabla f\|_{L^2} \to 0.$$

This proves $\widehat{\nabla f_m}$ converges in $L^2(\mathbb{R}^d)$ to $\widehat{\nabla f}$. But convergence in L^2 implies convergence in measure, i.e., $\forall \varepsilon > 0$

$$\lim_{n\to\infty} \left|\{K \in \mathbb{R}^d \text{ s.t. } |\widehat{\nabla f}_n(K) - \widehat{\nabla f}(K)| \geq \varepsilon\}\right| = 0.$$

Indeed, fix $\varepsilon > 0$. Then

$$\left|\{K \in \mathbb{R}^d \text{ s.t. } |\widehat{\nabla f}_n(K) - \widehat{\nabla f}(K)| \geq \varepsilon\}\right| = \int_{|\widehat{\nabla f}_n(K) - \widehat{\nabla f}(K)|^2 \geq \varepsilon^2} \frac{\varepsilon^2}{\varepsilon^2}\, dK$$

$$\leq \frac{1}{\varepsilon^2} \int_{|\widehat{\nabla f}_n(K) - \widehat{\nabla f}(K)|^2 \geq \varepsilon^2} |\widehat{\nabla f}_n(K) - \widehat{\nabla f}(K)|^2\, dK \leq \frac{1}{\varepsilon^2} \|\widehat{\nabla f}_n - \widehat{\nabla f}\|_{L^2} \xrightarrow[n\to\infty]{} 0.$$

But for any sequence converging in measure (with respect to a σ-finite measure) there exists a subsequence that converges pointwise almost everywhere (see e.g. Folland- Real Analysis Thm 2.30). Thus, there exists a subsequence $\widehat{\nabla f_{n_j}}(K) = 2\pi i K\, \hat{f}_{n_j}(K) \to \widehat{\nabla f}(K)$. K-almost everywhere. At the same time, again by the L^2-convergence and up to a further extraction of a subsequence, $\hat{f}_{n_j}(K) \to \hat{f}(K)$ K-almost everywhere, and therefore $2\pi i K \hat{f}_{n_j}(K) \to 2\pi i K \hat{f}(K)$ K-a.e.. This proves $\widehat{\nabla f}(K) = 2\pi i K \hat{f}(K)$, and therefore

$k \mapsto |k| \hat{f}(k)$ is in $L^2(\mathbb{R}^d)$ (since $\widehat{\nabla f}$ is, by assumption + Plancherel).
Assume now that $\hat{h}(k) = 2\pi i k \hat{f}(k)$ is in $L^2(\mathbb{R}^d)$. Denote by $h := \check{\hat{h}}$ and $\phi \in C_c^\infty(\mathbb{R}^d)$. Then

$$\int_{\mathbb{R}^d} \nabla\phi \, \bar{f} = \int_{\mathbb{R}^d} \widehat{\nabla\phi} \, \overline{\hat{f}} = 2\pi i \int_{\mathbb{R}^d} k \hat{\phi}(k) \, \overline{\hat{f}(k)} \, dk$$

again by the above formula valid on $S(\mathbb{R}^d)$ (or integration by parts).
But then

$$\int_{\mathbb{R}^d} \nabla\phi \, \bar{f} = - \int_{\mathbb{R}^d} \hat{\phi} \, \overline{\hat{h}} = - \int_{\mathbb{R}^d} \phi \, \bar{h}.$$

This proves that the distributional gradient of $\bar{f}$ is $\bar{h}$, which is in L^2. Hence $f \in H^1(\mathbb{R}^d)$. $\blacksquare$

A really important feature of Sobolev spaces is the existence of Sobolev inequalities. These are a way of controlling high L^q-norms in terms of $W^{k,p}$-norms, with $p < q$ and $k > 0$. Among their consequence are Sobolev embedding, i.e., inclusions of Sobolev spaces.
The following will be the most important Sobolev inequality for our purposes.

Theorem 3.10 (Sobolev inequality for gradients)

Let $d \geq 3$ and $f \in H^1(\mathbb{R}^d)$. Then $f \in L^q(\mathbb{R}^d)$ with $q = \frac{2d}{d-2}$. Moreover, there exists C_d such that

$$\|f\|_{L^q} \leq C_d \|\nabla f\|_{L^2} \quad \forall f \in H^1(\mathbb{R}^d).$$

Before proving the statement, a few remarks are in order.
First, notice that the value $q = \frac{2d}{d-2}$ is the only one for which the inequality has a chance of holding, as can be seen by the following scaling argument. Let $f \in C_c^\infty(\mathbb{R}^d)$, $k > 0$, and define the scaling

$$f_k(x) := f(kx).$$

This implies $\nabla f_K(x) = K(\nabla f)(Kx)$. By a change of variables we thus find

$$\|f_K\|_{L^q} = \left(\int_{\mathbb{R}^d} |f_K(x)|^q dx\right)^{1/q} = \left(\int_{\mathbb{R}^d} |f(Kx)|^q dx\right)^{1/q} = K^{-d/q} \|f\|_{L^q}$$

and

$$\|\nabla f_K\|_{L^2} = \left(\int_{\mathbb{R}^d} |\nabla f_K(x)|^2 dx\right)^{\frac{1}{2}} = K^{1-\frac{d}{2}} \|\nabla f\|_{L^2}.$$

Hence, $\|f_K\|_{L^q} \leq C_d \|\nabla f_K\|_{L^2}$ is equivalent to

$$\|f\|_{L^q} \leq C_d K^{1-\frac{d}{2}+\frac{d}{q}} \|\nabla f\|_{L^2}$$

Since K can be arbitrarily small or large, the inequality can only hold if

$$1-\frac{d}{2}+\frac{d}{q} = 0 \iff q = \frac{2d}{d-2}.$$

This of course does not prove that the inequality holds.

We further remark that the inequality has a version for $W^{1,p}(\mathbb{R}^d)$ instead of $H^1(\mathbb{R}^d)$: for every $1 \leq p < d$, there exists $C_{d,p}$ such that

$$\|f\|_{L^q} \leq C_{d,p} \|\nabla f\|_{L^p} \quad \forall f \in W^{1,p}(\mathbb{R}^d),$$

with $q = \dfrac{dp}{d-p}$. This implies in particular $W^{1,p}(\mathbb{R}^d) \subset L^q(\mathbb{R}^d)$ with such a q.

In the proof of Theorem 3.10 we will need the following important result, which we state without proof.

Theorem 3.11 (Hardy-Littlewood-Sobolev inequality)

Let $p,r>1$, and $0<\lambda<d$ with $\frac{1}{p}+\frac{\lambda}{d}+\frac{1}{r}=2$. Then there exists a constant C (depending on p,λ,d) such that

$$\left|\int_{\mathbb{R}^d\times\mathbb{R}^d} f(x)\frac{1}{|x-y|^\lambda} h(y)\,dx\,dy\right| \leq C\|f\|_{L^p}\,\|h\|_{L^r}$$

for all $f \in L^p(\mathbb{R}^d)$, $h \in L^r(\mathbb{R}^d)$.

The Hardy-Little-Sobovev inequality is sometimes called 'weak Young's inequality'. It is very much like a Young(+Hölder)-type inequality

$$\| f (\tfrac{1}{|\cdot|^\lambda} * h) \|_{L^1} \leq \|f\|_{L^p} \|h\|_{L^r} \cdot \| \tfrac{1}{|\cdot|^\lambda} \|_{L^{d/\lambda}} .$$

The problem here is that $\frac{1}{|\cdot|^\lambda}$ does not belong to any L^p-space for $1 \leq p < +\infty$ (it however belongs to the "weak $L^{d/\lambda}_w(\mathbb{R}^d)$ space", whatever that is, hence the name). Theorem 3.11 states that, even though $\frac{1}{|\cdot|^\lambda} \notin L^{d/\lambda}(\mathbb{R}^d)$, a Young-type inequality still holds. Notice that, even thoug, as said, $1/|\cdot|^\lambda \notin L^{d/\lambda}(\mathbb{R}^d)$, this is due to mild divergencies:

$$\int_{|x|<\varepsilon} \left| \frac{1}{|x|^\lambda} \right|^{d/\lambda} dx \leq C \int_0^\varepsilon \frac{|r|^{d-1}}{|r|^d} dr = c \log(\tfrac{1}{\varepsilon})$$

$$\int_{|x|>\frac{1}{\varepsilon}} \left| \frac{1}{|x|^\lambda} \right|^{d/\lambda} \leq C \log(\tfrac{1}{\varepsilon}) .$$

So $\frac{1}{|\cdot|^\lambda}$ "almost" belongs to $L^{d/\lambda}(\mathbb{R}^d)$.

We are now ready to prove Theorem 3.10.

Proof of Theorem 3.10

Recall the definition

$$G_d(x) = \frac{1}{(d-2)|S_{d-1}|} \frac{1}{|x|^{d-2}}$$

for the Green's function of the Laplace operator.

We first claim the estimate

$$|\langle f, g \rangle| \leq C \|\nabla f\|_{L^2} |\langle g, G_d * g \rangle|^{1/2} \qquad (*)$$

for every $f \in \dot{H}^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ and $g \in L^{q'}(\mathbb{R}^d)$ with $\frac{1}{q}+\frac{1}{q'}=1$.
Recall that, by definition,

$$\langle f,g\rangle = \int_{\mathbb{R}^d} \overline{f(x)}\, g(x)\,dx.$$

We first prove ⊛ for $f \in \dot{H}^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ and $g \in C_c^\infty(\mathbb{R}^d)$. We have

$$
\begin{aligned}
|\langle f,g\rangle| = |\langle \hat f,\hat g\rangle| &= \left| \int_{\mathbb{R}^d} \overline{\hat f(k)}\, \hat g(k)\,dk \right| \\
&\le \left(\int_{\mathbb{R}^d} |k|^2\, |\hat f(k)|^2\,dk\right)^{\frac{1}{2}} \cdot \left(\int_{\mathbb{R}^d} \frac{1}{|k|^2}\, |\hat g(k)|^2\,dk\right)^{\frac{1}{2}}, \\
&= \|\nabla f\|_{L^2} \left(\int \frac{1}{|k|^2}\, |\hat g(k)|^2\,dk\right)^{\frac{1}{2}}
\end{aligned}
$$

having used Parseval's identity and Hölder's inequality.
Next, we remark the identity

$$\left(\frac{1}{|\cdot|}\hat g\right)^{\vee}(x) = C \int_{\mathbb{R}^d} \frac{1}{|x-y|^{d-1}}\, g(y)\,dy, \quad \text{for appropriate } C>0.$$

The proof of this is left as exercise. It essentially follows from the fact that the Fourier transform maps products to convolutions (Theorem 1.9) and the formula (see Exercise 5, sheet 2)

$$\widehat{|\cdot|^{\alpha-d}} = c\,|\cdot|^{-\alpha} \quad \text{for appropriate } c>0.$$

We thus apply (twice) the above identity and deduce

$$
\begin{aligned}
|\langle f,g\rangle| &\le \|\nabla f\|_{L^2} \cdot \left[\int_{\mathbb{R}^d} \left| \left(\frac{1}{|\cdot|}\hat g(\cdot)\right)^{\vee}(x) \right|^2 dx \right]^{\frac{1}{2}} \\
&= C \|\nabla f\|_{L^2} \left[\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{1}{|x-y|^{d-1}}\, g(y)\,dy \right)\left(\int_{\mathbb{R}^d} \frac{1}{|x-z|^{d-1}}\, g(z)\,dz \right) dx \right]^{\frac{1}{2}} \\
&= C \|\nabla f\|_{L^2} \left[\int_{\mathbb{R}^d \times \mathbb{R}^d} g(y)\,g(z) \left(\int_{\mathbb{R}^d} \frac{1}{|x-y|^{d-1}}\, \frac{1}{|x-z|^{d-1}}\,dx \right) dy\,dz \right]^{\frac{1}{2}}
\end{aligned}
$$

$$= C \|\nabla f\|_{L^2} \langle g, G_d * g \rangle^{\frac{1}{2}},$$

where the last line follows from the equality

$$\int_{\mathbb{R}^d} \frac{1}{|x-y|^{d-1}} \frac{1}{|x-z|^{d-1}} \, dx = C \frac{1}{|y-z|^{d-1}} \quad \text{for appropriate } C>0,$$

whose proof is left as exercise. We have thus proven ⊛ for $f \in H^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ and $g \in C_c^\infty(\mathbb{R}^d)$. Suppose now $g \in L^{q'}(\mathbb{R}^d)$ with $1/q + 1/q' = 1$, as in the assumptions. We conclude the proof of ⊛ by a standard density argument. There exists a sequence $g_n \in C_c^\infty(\mathbb{R}^d)$ such that $\|g_n - g\|_{L^{q'}} \to 0$ as $n \to \infty$ (Theorem 0.7). Since

$$|\langle f, g \rangle - \langle f, g_n \rangle| = \left| \int_{\mathbb{R}^d} \overline{f(x)} \, (g(x) - g_n(x)) \, dx \right|$$

$$\leq \|f\|_{L^q} \, \|g - g_n\|_{L^{q'}}$$

by Hölder's inequality, we have $\langle f, g \rangle = \lim_{n\to\infty} \langle f, g_n \rangle$.
Recall now the Hardy-Littlewood-Sobolev inequality (Theorem 3.11). Choosing there $\lambda = d-2$, we would have the relation $\frac{1}{p} + \frac{1}{r} = 1 + \frac{2}{d}$ for the indexes in Theorem 3.11. Now, since $q = \frac{2d}{d-2}$ and $\frac{1}{q} + \frac{1}{q'} = 1$, we have $q' = \frac{2d}{d+2}$.
We can thus apply HLS for $p = r = q'$:

$$|\langle h_1, G_d * h_2 \rangle| = C \left| \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{|x-y|^{d-2}} \overline{h_1(x)} \, h_2(y) \, dx dy \right| \leq C \|h_1\|_{L^{q'}} \|h_2\|_{L^{q'}}.$$

This implies

$$|\langle g_n, G_d g_n \rangle - \langle g, G_d g \rangle| \leq |\langle (g_n - g), G_d g_n \rangle| + |\langle g, G_d (g - g_n) \rangle|$$

$$\leq \|g_n - g\|_{L^{q'}} \left(\|g\|_{L^{q'}} + \|g_n\|_{L^{q'}} \right) \xrightarrow[n\to\infty]{} 0.$$

This shows $\lim_{n\to\infty} \langle g_n, G_d g_n \rangle = \langle g, G_d g \rangle$, and therefore, since ⊛ holds for g_n, we have

$$|\langle f,g\rangle| = \lim_{n\to\infty} |\langle f,g_n\rangle| \leq C\|\nabla f\|_{L^2} \lim_{n\to\infty} \langle g_n, G_d\, g_n\rangle^{1/2}$$

$$= C\|\nabla f\|_{L^2} \langle g, G_d\, g\rangle^{1/2}.$$

This completes the proof of ⊛.

To prove the inequality in the statement in the case $f\in H^1(\mathbb{R}^d)\cap L^q(\mathbb{R}^d)$, we argue by duality:

$$\|f\|_{L^q} = \sup_{\substack{g\in L^{q'}(\mathbb{R}^d)\\ \|g\|_{L^{q'}}\leq 1}} |\langle f,g\rangle| \leq C\|\nabla f\|_{L^2} \sup_{\substack{g\in L^{q'}(\mathbb{R}^d)\\ \|g\|_{L^{q'}}\leq 1}} \langle g, G_d\, g\rangle^{1/2}$$

$$\leq C\|\nabla f\|_{L^2} \sup_{\substack{g\in L^{q'}(\mathbb{R}^d)\\ \|g\|_{L^{q'}}\leq 1}} \|g\|_{L^{q'}}^2$$

$$= C\|\nabla f\|_{L^2}.$$

This proves the claim for $f\in H^1(\mathbb{R}^d)\cap L^q(\mathbb{R}^d)$.

Suppose now $f\in H^1(\mathbb{R}^d)$ arbitrary. Let $\chi\in C_c^\infty([0,\infty))$ decreasing, with $\chi(t)=1$ for $t<1$ and $\chi(t)=0$ for $t>2$. Define now the truncation

$$f_n(x) = \begin{cases} f(x) & |f(x)|\leq n \\ n & |f(x)|>n. \end{cases}$$

Then

$$\nabla f_n(x) = \begin{cases} \nabla f(x) & |f(x)|<n \\ 0 & |f(x)|=n \leftarrow \text{exercise 1 sheet 4} \\ 0 & |f(x)|>n. \end{cases}$$

Since $f_n\in L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ we have $f_n\in L^p(\mathbb{R}^d)$ $\forall\, 2\leq p<\infty$, and in particular $p=q$. Moreover, $f_n\in H^1(\mathbb{R}^d)$ since

$$\int_{\mathbb{R}^d} |\nabla f_n|^2 \leq \int_{\mathbb{R}^d} |\nabla f|^2 .$$

By monotone convergence we also find $\|f_n\|_{L^q}\to\|f\|_{L^q}$ (possibly infintite) and $\|\nabla f_n\|_{L^2}\to\|\nabla f\|_{L^2}$ (finite by assumption).

Since, by what we proved above for $H^1(\mathbb{R}^d)\cap L^q(\mathbb{R}^d)$ - functions,

$$\|f_n\|_{L^q} \leq C\|\nabla f_n\|_{L^2},$$

we deduce that the inequality must remain valid in the limit. Since the right hand side converges to a finite number, the left hand side must converge as well, and therefore $f\in L^q(\mathbb{R}^d)$. ■

We remark again that $\|\nabla f\|_2$ can only control $\|f\|_{L^q}$ with $q=\frac{2d}{1-2}$. However, if $f\in H^1(\mathbb{R}^d)$ it of course belongs to $L^2(\mathbb{R}^d)$, and therefore, if $2\leq p\leq q$, an interpolation shows

$$\|f\|_{L^p} \leq \|f\|_{L^2}^{\alpha_p}\, \|f\|_{L^q}^{1-\alpha_p}$$

for a suitable $0\leq \alpha_p\leq 1$. This implies

$$\|f\|_{L^p}\leq C\|f\|_{L^2}^{\alpha_p}\,\|\nabla f\|_{L^2}^{1-\alpha_p} \leq C\left(\|f\|_{L^2}+\|\nabla f\|_{L^2}\right)=C\|f\|_{H^1}$$

$ab\leq \frac{a^\alpha}{\alpha}+\frac{b^\beta}{\beta}$ if $\alpha,\beta>1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$

This shows that the H^1-norm controls the L^p norm for all $2\leq p\leq q=\frac{2d}{d-2}$ if $d\geq 3$.

For $d=1,2$ the situation is slightly different.

Theorem 3.12 (Sobolev's inequality for gradients, $d=1,2$)

There exists $C>0$ such that

$$\|f\|_{L^q} \leq C\|f\|_{H^1} \qquad \forall f\in H^1(\mathbb{R})\ \forall\, 2\leq q\leq\infty.$$

Moreover, $f\in H^1(\mathbb{R})$ has a representative such that

$$|f(x)-f(y)| \leq \|\partial f\|_{L^2}\,|x-y|^{1/2},$$

i.e., $H^1(\mathbb{R})$ functions are Hölder continuous with exponent $1/2$.

Moreover, for every $2 \le q < \infty$ $\exists\, C_q$ such that

$$\|f\|_{L^q} \le C_q \|f\|_{H^1} \qquad \forall f \in H^1(\mathbb{R}^2).$$

We can thus write the recap

$$H^1(\mathbb{R}^d) \subset L^q(\mathbb{R}^d) \qquad \begin{cases} \forall\ 2 \le q \le \infty & d=1 \\ \forall\ 2 \le q < \infty & d=2 \\ \forall\ 2 \le q \le \frac{2d}{d-2} & d \ge 3. \end{cases}$$

Notice that there are explicit counterexamples for the embedding $H^1(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2)$.

What happens for, say, $H^1(\Omega)$ instead of $H^1(\mathbb{R}^d)$, with $\Omega \subset \mathbb{R}^d$ open?

First, if $f \in H^1(\Omega)$ is compactly supported, then we can extend it by continuing it as 0 on $\mathbb{R}^d / \operatorname{supp} f$. Then, if $\tilde{f}$ is such an extension

$$\|f\|_{L^q(\Omega)} = \|\tilde{f}\|_{L^q(\mathbb{R}^d)} \le C \|\tilde{f}\|_{H^1(\mathbb{R}^d)} = C \|f\|_{H^1(\mathbb{R}^d)}.$$

for all the q's discussed above ($2 \le q \le \frac{2d}{d-2}$ if $d \ge 3$, $2 \le q < \infty$ if $d=2$ and $2 \le q \le \infty$ if $d=1$).

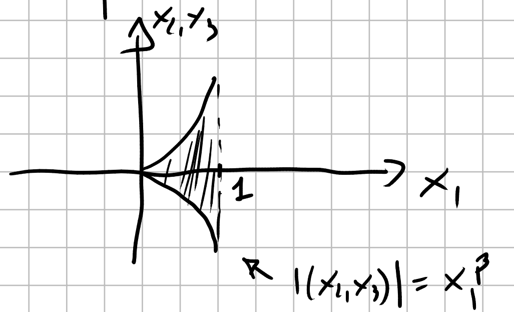
This is true (by density) even if f is not compactly supported, but it belongs to $H^1_0(\Omega)$, which is the completion of $C^\infty_0(\Omega)$ with respect to the Sobolev norm $\|\cdot\|_{H^1(\Omega)}$. Since, for general $\Omega \subset \mathbb{R}^d$ open, $H^1_0(\Omega) \subsetneq H^1(\Omega)$, we still don't know whether the inequality extends to generic $f \in H^1(\Omega)$.

The answer depends on the "form" of Ω.

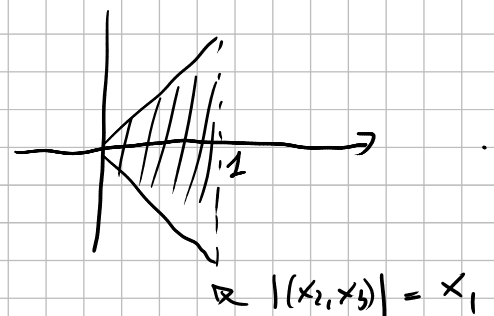
Example For $\beta \ge 1$, consider the set

$$\Omega_\beta = \{ x \in \mathbb{R}^3 \text{ s.t. } 0 < x_1 < 1,\ (x_2^2 + x_3^2)^{\frac{1}{2}} < x_1^{\beta} \}.$$

For $\beta > 1$



For $\beta = 1$



Consider the function $f(x) = |x|^{-\alpha}$ for some $\alpha > 0$, for which $|\nabla f(x)| = \alpha |x|^{\alpha - 1}$.

By a not too hard calculation, (Benjamin Schlein's notes)

$$\int_{\Omega_\beta} |\nabla f(x)|^2 dx = C \int_0^1 dx_1 \frac{1}{x_1^{2\alpha}} \Bigg[\underbrace{1 - \frac{1}{(1 + x_1^{2(\beta-1)})^\alpha}}_{\simeq \alpha x_1^{2(\beta-1)} \text{ as } x_1 \simeq 0} \Bigg]$$

and therefore the integral is finite iff $2\alpha - 2(\beta-1) < 1$ i.e. $\alpha < \beta - \frac{1}{2}$. On the other hand,

$$\int_{\Omega_\beta} |f(x)|^6 dx = C \int_0^1 dx_1 \frac{1}{x_1^{6\alpha-2}} \left[1 - \frac{1}{(1 + x_1^{2(\beta-1)})^{3\alpha-1}} \right],$$

which is finite iff $6\alpha - 2 - 2(\beta - 1) < 1$ i.e. $\alpha < \beta/3 + \frac{1}{6}$.

The inequality

$$\|f\|_{L^6(\Omega_\beta)} \leq C \|\nabla f\|_{L^2(\Omega_\beta)}$$

can't be true if $\exists \alpha$ contradicting the second condition but verifying the first one, i.e.

$$\beta/3 + \frac{1}{6} < \alpha < \beta - \frac{1}{2}$$

This is always possible if $\beta > 1$.

We find a contradiction to the validity of Sobolev's inequalities on "horns" but no contradictions in the case in the case $\beta=1$, for which Ω_β is a cone. It turns out that cones are good domains for Sobolev's inequalities.

Definition 3.13 (Cone property)

Let

$$K_{r,\theta} = \left\{ x \in \mathbb{R}^n \mid x \neq 0, \quad 0 < x_n < |x| \cos\theta \right\} \cap B_r(0).$$

We say that $\Omega \subseteq \mathbb{R}^d$ open has the cone property if, $\exists\, r, \theta$ such that, $\forall x \in \Omega$, $\exists$ a cone K_x congruent to $K_{r,\theta}$, with vertex at x and entirely contained in Ω.

Notice that, e.g., balls, cones, cylinders have the cone property. Horns do not.

Theorem 3.14 (General Sobolev inequalities)

Let $\Omega \subset \mathbb{R}^d$ be open and with the cone property for some r, θ. Let $1 \leq p \leq q \leq \infty$, $m, k \in \mathbb{N}$, $k \leq m$. Then $\exists\, C = C_{m,n,p,q,r,\theta}$ such that, $\forall f \in W^{k,p}(\Omega)$:

a) If $kp < d$, then $\|f\|_{W^{m-k,q}(\Omega)} \leq C \|f\|_{W^{m,p}(\Omega)}$

for all $p \leq q \leq \dfrac{dp}{d-kp}$.

b) If $kp = d$, then $\|f\|_{W^{m-k,q}(\Omega)} \leq C \|f\|_{W^{m,p}(\Omega)}$

for all $p \leq q < \infty$.

c) If $kp > d$, then $\|f\|_{W^{m-k,q}(\Omega)} \leq C \|f\|_{W^{m,p}(\Omega)}$

for all $p \leq q \leq \infty$.

k is the number of derivatives we want to throw away in order to bound low $W^{m-k,q}$-norms with high $W^{m,p}$-norms. In our concrete cases above, $m=k=1$ and $p=2$.

We will get back to Sobolev's embedding after the chapter on weak convergence, to discuss compact embeddings.