2 Distributions and distributional derivative

Distributions are a very important tool in the study of differential equations. From the technical point of view they allow to completely neglect an amount of subtle questions about differentiability, since every locally integrable function can be distributionally differentiated an arbitrary number of times.
For a more physics-oriented motivation, let us consider the non-homogeneous Laplace equation

$$-\Delta u = f \qquad \text{on } \mathbb{R}^d.$$

When f has the interpretation of, e.g., electric charge density or mass density, then the solution u has the interpretation of electric or gravitational potential.
When f is regular, it models for example an object with finite size.
But how to model a point charge or point mass? The corresponding f should be $=0$ everywhere except for at one point, where it must be equal to... something else. But $f=0$ a.e. means $f=0$ in every L^p, so no (locally) L^p function can model a point object. Distributions (in this specific case the Dirac delta distribution) provide a way to implement this.

Distributions are, vaguely speaking, objects defined by their actions on test functions. Let us precisely state what we mean by the latter.

Definition 2.1 (The space $D(\Omega)$)

Let $\Omega \subseteq \mathbb{R}^d$ be open and nonempty. The space of test functions $D(\Omega)$ consists of all functions in $C_c^\infty(\Omega)$ (smooth and compactly supported functions) with the following notion of convergence:

a sequence $\phi_m \in C_c^\infty(\Omega)$ converges in $D(\Omega)$ to $\phi \in C_c^\infty(\Omega)$ if and only if $\exists$ a fixed compact set $K \subset \Omega$ such that

$$\operatorname{supp}(\phi_m - \phi) \subseteq K \quad \forall m \qquad \text{and} \qquad \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d} \phi_m \xrightarrow[m\to\infty]{} \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d} \phi$$

for every integers $\alpha_1, \dots, \alpha_d \geq 0$ uniformly on K.

Recall that $\psi_m \to \psi$ uniformly on K if

$$\sup_{x \in K} |\psi_m(x) - \psi(x)| \to 0 \quad \text{as } m \to \infty.$$

Roughly speaking, $D(\Omega)$ is the set $C_c^\infty(\Omega)$ but in which we say that sequences converge only if they behave very nicely: they have to converge uniformly with all their derivatives and they have to do so by staying in a fixed compact set. For example, a sequence of the type

does not converge to zero in $D(\Omega)$ even though it converges uniformly to zero.

Definition 2.2 (Distribution)

A distribution T is a continuous linear functional on $\mathcal{D}(\Omega)$, that is, $T : \mathcal{D}(\Omega) \to \mathbb{C}$ such that

$$T(\phi_1 + \phi_2) = T(\phi_1) + T(\phi_2) \qquad T(\lambda\phi) = \lambda T\phi$$

$\forall \phi, \phi_1, \phi_2 \in \mathcal{D}(\Omega)$, $\lambda \in \mathbb{C}$. Continuity means that, for any sequence ϕ_n in $\mathcal{D}(\Omega)$ with $\phi_n \to \phi \in \mathcal{D}(\Omega)$ as $n \to \infty$, we have

$$T(\phi_n) \to T(\phi) \quad \text{as } n \to \infty.$$

The space of distributions is denoted by $\mathcal{D}'(\Omega)$, the dual of $\mathcal{D}(\Omega)$. Continuity is a rather weak requirement: the convergence notion in $\mathcal{D}(\Omega)$ is very strong (very few sequences converge), so the fact that T must satisfy $T(\phi_n) \to T(\phi)$ for any convergent $\phi_n \to \phi$, is not a strong constraint.

Definition 2.3 (Convergence of distributions)

A sequence of distributions $T_n \in \mathcal{D}'(\Omega)$ converges in $\mathcal{D}'(\Omega)$ to $T \in \mathcal{D}'(\Omega)$ if, $\forall \phi \in \mathcal{D}(\Omega)$,

$$\lim_{n\to\infty} T_n(\phi) = T(\phi).$$

We will see that, again, this is a very weak notion of convergence (e.g. derivatives of a converging sequence of distributions converge automatically).

What are, concretely, distributions? What does it mean to be a distribution? The most important example of distribution is... a (locally integrable) function!
To discuss this, we will need the following generalization of L^p spaces.

Definition 2.4 ($L^p_{loc}(\Omega)$)

$L^p_{loc}(\Omega)$ is the space of Lebesgue measurable functions defined on Ω such that

$$\|f\|_{L^p(K)} = \left(\int_K |f(x)|^p dx\right)^{1/p} < +\infty$$

for every compact $K \subset \Omega$.

A sequence $f_n \in L^p_{loc}(\Omega)$ converges to $f \in L^p_{loc}(\Omega)$ if $f_n \to f$ in $L^p(K)$ for every compact $K \subset \Omega$, i.e.,

$$\lim_{n\to\infty} \int_K |f_n(x) - f(x)|^p dx = 0 \qquad \forall K \subset \Omega \text{ compact.}$$

A few remaks are in order.

- $f \in L^p_{loc}(\Omega)$ does not imply $f \in L^p(\Omega)$. For example, $f(x) = x$ belongs to $L^p_{loc}(\mathbb{R})$ for every $1 \le p \le +\infty$, but to no $L^p(\mathbb{R})$.
- $L^p_{loc}(\Omega) \supset L^r_{loc}(\Omega)$ for $r > p$, since these spaces are only sensible to local singularities (and not to behavior at ∞), and increasing the p index allows to consider less and less singular functions.
  Notice that, in turn, $L^p(\Omega) \supset L^r(\Omega)$ for $r > p$ is <u>false</u> (unless Ω has finite measure) since decay at infinity matters.

$L^1_{loc}(\Omega)$ is the most important space when talking about distributions. Let $f \in L^1_{loc}(\Omega)$. For any test-function $\phi \in \mathcal{D}(\Omega)$, the expression

$$T_f(\phi) := \int_\Omega f(x)\phi(x)dx$$

defines a linear functional on $\mathcal{D}(\Omega)$. This defines a distribution if it is continuous in the sense of Definition 2.2. Let's check it.
Let $\phi_n \in \mathcal{D}(\Omega)$ be a sequence converging in $\mathcal{D}(\Omega)$ to some $\phi \in \mathcal{D}(\Omega)$. Then

$$|T_f(\phi) - T_f(\phi_n)| = \left| \int_\Omega (\phi(x) - \phi_n(x)) f(x) dx \right|$$

$$= \left| \int_K (\phi(x) - \phi_n(x)) f(x) dx \right|$$

$$\leq \underbrace{\sup_{x \in K} |\phi(x) - \phi_n(x)|}_{\xrightarrow{n\to\infty} 0} \int_K |f(x)| dx$$

where the compact set $K \subset \Omega$ exists by Definition 2.1. The $\sup_{x\in K}|\dots|$ tends to zero again by Definition 2.1, while $\int_K |f|$ is a finite, fixed number, since $f \in L^1_{loc}(\Omega)$.

We say that a distribution T is the function f if T is of the type T_f for some $f \in L^1_{loc}(\Omega)$.
Are these the only existing distributions? Certainly not! An important example is the Dirac delta distribution. Consider the map on $\mathcal{D}(\Omega)$

$$\delta_x : \mathcal{D}(\Omega) \to \mathbb{C} \quad \text{defined by} \quad \delta_x(\phi) = \phi(x)$$

where $x \in \Omega$ is fixed. We leave as exercise to prove that $\delta_x \in \mathcal{D}'(\Omega)$, and that it is not of the type T_f for $f \in L^1_{loc}(\Omega)$ (using also the next result)

Theorem 2.5 (Functions in L^1_{loc} are uniquely determined by distributions)

Let $\Omega \subseteq \mathbb{R}^d$ be open, $f, g \in L^1_{loc}(\Omega)$. Suppose $T_f = T_g$, i.e.,

$$\int_\Omega f\phi = \int_\Omega g\phi \qquad \forall \phi \in \mathcal{D}(\Omega).$$

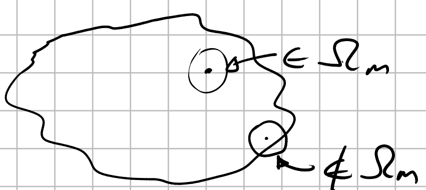
Then $f(x) = g(x)$ for almost every $x \in \Omega$.

The main message of this result is that functions in $L^1_{loc}(\Omega)$ and their associated distributions are (different but) identifiable objects.

Proof

For $m = 1, 2, \ldots$, let

$$\Omega_m = \{ x \in \Omega \text{ such that } x + y \in \Omega \text{ for } |y| \leq \tfrac{1}{m} \}.$$



It is easy to see that Ω_m is open. Let $J \in C_c^\infty(\mathbb{R}^d)$ with supp J included in the unit ball and $\int_{\mathbb{R}^n} J = 1$. Define the rescaling $J_m(x) = m^d J(mx)$ with the properties $\int J_m = 1$ and supp J_m included in the ball of radius m. Fix M. If $m \geq M$, then using the assumption $T_f(\phi) = T_g(\phi)$ with $\phi(y) = J_m(x-y)$ for any fixed $x \in \Omega_M$, we have

$$(J_m * f)(x) = \int_\Omega J_m(x-y) f(y) dy = \int_\Omega J_m(x-y) g(y) dy = (J_m * g)(x).$$

By the approximation theorem with C^∞ functions, $J_m * f \xrightarrow{m\to\infty} f$ and $J_m * g \xrightarrow{m\to\infty} g$ in $L^1_{loc}(\Omega_M)$. Thus $f = g$ in $L^1_{loc}(\Omega_M)$, which implies $f(x) = g(x)$ for almost every $x \in \Omega_M$. Sending now $M \to \infty$ allows to cover (almost) every $x \in \Omega$. ∎

The above result implies the following form of the fundamental lemma of calculus of variations:

Let $f \in L^1_{loc}(\Omega)$ such that $\int_\Omega f\phi = 0$ for every $\phi \in \mathcal{D}(\Omega)$. Then $f = 0$ almost everywhere in Ω.

We now address the problem of defining derivatives of distributions.

Definition 2.6 (Distributional or weak derivative)

Let $T \in \mathcal{D}'(\Omega)$ and $\alpha_1, \dots, \alpha_d$ be non-negative integers. Define the distribution

$$\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d} T \equiv D^\alpha T \qquad \text{with} \quad \alpha = (\alpha_1, \dots, \alpha_d)$$

by its action on $\phi \in \mathcal{D}(\Omega)$ as follows:

$$(D^\alpha T)(\phi) = (-1)^{|\alpha|} T(D^\alpha \phi)$$

where $|\alpha| = \sum_{j=1}^{d} |\alpha_j|$.

In the special case $\alpha_i = 1$, $\alpha_j = 0$ $j \neq i$, we use the symbol $\partial_i T$ for $D^\alpha T$. Moreover, we call $\nabla T := (\partial_1 T, \dots, \partial_n T)$ the distributional gradient.

The next result, whose proof we leave as exercise, shows that the above definition is well posed.

Proposition 2.7

1) For any $T \in \mathcal{D}'(\Omega)$, $D^\alpha T$ is a continuous linear map on $\mathcal{D}(\Omega)$, and it thus indeed defines a distribution.
2) Differentiation is a continuous operation: if $T_n \to T$ in $\mathcal{D}'(\Omega)$, as $n \to \infty$, then $D^\alpha T_n \to D^\alpha T$ in $\mathcal{D}'(\Omega)$.

How do derivatives of distributions look like? Let's consider the case of distributions of the type T_f. For $f \in C^{|\alpha|}(\Omega)$, we have

$$\begin{aligned}(D^\alpha T_f)\phi &= (-1)^{|\alpha|} T_f(D^\alpha \phi) \\ &= (-1)^{|\alpha|} \int_\Omega f(x)\,(D^\alpha \phi)(x)\,dx \\ &= \int_\Omega (D^\alpha f)(x)\,\phi(x)\,dx = T_{D^\alpha f}(\phi),\end{aligned}$$

where the third equality holds by integration by parts (no boundary term, since ϕ has compact support).

Hence, the distributional derivative agrees with the classical notion of derivative when the latter exists and is continuous. In fact, the following stronger result could be proven.

Theorem 2.8 (Equivalence between classical and distributional derivative)

Let $\Omega \subseteq \mathbb{R}^d$ open. Let $T \in \mathcal{D}'(\Omega)$, and define $G_i := \partial_i T \in \mathcal{D}'(\Omega)$ for $i=1,..,d$. The following are equivalent:

1) T is a function $f \in C^1(\Omega)$ (up to a set of zero measure).

2) G_i is a function $g_i \in C^0(\Omega)$ for each $i=1,..d$.

In each case, $g_i = \frac{\partial f}{\partial x_i}$, the classical derivative.

The distributional derivative is however a strong extension since every distribution is differentiable infinitely many times. In particular, every (non-smooth) function is differentiable infinitely many times in the distributional sense. The distributional derivatives, however, are not necessarily functions.

This is one of the standard ways of constructing distributions that are not functions. Unless $f \in C^\infty$, for $|\alpha|$ large enough $D^\alpha T_f$ will stop being of the form $T_{D^\alpha f}$. We will see in exercise class that, for the continuous function $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} x & x \geq 0 \\ 0 & x < 0 \end{cases}$$

one has $DT_f = T_{Df}$, where $Df = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$ (defined almost everywhere), while $D^2 T_f = \delta_0$, Dirac delta in $x=0$.

We have seen that $L^1_{loc}(\Omega)$-functions naturally give rise to distributions. Refining the space $L^1_{loc}(\Omega)$ by allowing functions to have distributional derivative in $L^1_{loc}(\Omega)$ naturally leads to the first definition of Sobolev space (we present the generic L^p_{loc}-case).

Definition 2.9 ($W^{1,p}_{loc}(\Omega)$ and $W^{1,p}(\Omega)$)

For each $1 \le p \le \infty$, we define

$$W^{1,p}_{loc}(\Omega) := \{ f : \Omega \to \mathbb{C} \text{ such that } f \in L^p_{loc}(\Omega) \text{ and } \partial_i f, \text{ as distribution in } \mathcal{D}'(\Omega), \text{ is an } L^p_{loc}(\Omega) \text{ function for each } i = 1, \dots, d \}.$$

$$W^{1,p}(\Omega) := \{ f : \Omega \to \mathbb{C} \text{ such that } f, \partial_i f \in L^p(\Omega) \text{ for } i = 1, \dots, d \}.$$

$W^{1,p}(\Omega)$ is a normed space with the norm

$$\|f\|_{W^{1,p}(\Omega)} := \left\{ \|f\|^p_{L^p(\Omega)} + \sum_{j=1}^{n} \|\partial_i f\|^p_{L^p(\Omega)} \right\}^{\frac{1}{p}}$$

The spaces $W^{1,p}(\Omega)$ are called Sobolev spaces. One always has to keep in mind that the notation $W^{m,p}(\Omega)$ denotes "space of functions in L^p with derivatives up to m-th order in L^p".
We will study them in more detail in the next chapter.

We now pass to the discussion of tempered distributions, a particular subspace of $\mathcal{D}'(\mathbb{R}^d)$ enjoying nice properties. In particular, they have a naturally well-defined Fourier transform.
First, we recall Parseval's identity

$$\langle f, g \rangle_{L^2} \equiv \int_{\mathbb{R}^d} \overline{f(x)}\, g(x)\, dx = \int \overline{\hat{f}(x)}\, \hat{g}(x)\, dx \equiv \langle \hat{f}, \hat{g} \rangle_{L^2}$$

which holds for any $f, g \in L^2(\mathbb{R}^d)$. Now, if $\varphi \in \mathcal{D}(\mathbb{R}^d)$ is a test function, we certainly have $\varphi \in L^2(\mathbb{R}^d)$. Then, if $f \in L^2(\mathbb{R}^d)$, then

$$\int_{\mathbb{R}^d} f(x)\, \hat{\varphi}(x) = \int_{\mathbb{R}^d} \overline{\overline{f(x)}}\, \hat{\varphi}(x)\, dx = \int_{\mathbb{R}^d} \overline{\hat{\overline{f}}(k)}\, \hat{\hat{\varphi}}(k)\, dk$$

$$= \int_{\mathbb{R}^d} \hat{f}(-k)\, \varphi(-k)\, dk = \int_{\mathbb{R}^d} \hat{f}(k)\, \varphi(k)\, dk$$

Hence, if T_f is the distribution associated to f, we would be tempted to write

$$\hat{T_f}(\varphi) = T_f(\hat{\varphi}).$$

and extend this to a definition of FT for generic distributions. The problem is that $\hat{\varphi}$ is not necessarily compactly supported, and therefore not in $\mathcal{D}(\Omega)$. So the definition wouldn't make sense. We need to enlarge the space of test functions.

Definition 2.10 (Schwarz space)

Let $\varphi \in C^\infty(\mathbb{R}^d)$. We define, for every $\alpha, \beta \in \mathbb{N}^d$,

$$\|\varphi\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^d} |x^\beta D^\alpha \varphi(x)|.$$

We say that $\varphi \in S(\mathbb{R}^d)$, the Schwarz space of rapidly decaying functions, if and only if $\|\varphi\|_{\alpha,\beta} < +\infty \quad \forall \alpha, \beta \in \mathbb{N}^d$.
We also say that a sequence $\varphi_n \in S(\mathbb{R}^d)$ converges to $\varphi \in S(\mathbb{R}^d)$ in $S(\mathbb{R}^d)$ if and only if

$$\lim_{n \to \infty} \|\varphi_n - \varphi\|_{\alpha,\beta} = 0 \quad \forall \alpha, \beta \in \mathbb{N}^d.$$

This space is slightly larger than $\mathcal{D}(\mathbb{R}^d)$. It contains functions that decay at ∞ faster than every inverse power, together with their derivatives. The typical example is $\varphi(x) = e^{-x^2}$.

The Fourier transform of functions in $S(\mathbb{R}^d)$ enjoys the following nice properties.

Proposition 2.11 (Fourier transform in $S(\mathbb{R}^d)$)

1) For every $\varphi \in S(\mathbb{R}^d)$ and $\alpha, \beta \in \mathbb{N}^d$ we have

$$\widehat{(D^\alpha \varphi)}(k) = (2\pi i)^{|\alpha|} k^\alpha \hat{\varphi}(k).$$

2) For every $\varphi \in S(\mathbb{R}^d)$, $\hat{\varphi} \in S(\mathbb{R}^d)$.

Proof

We prove 1) for $\alpha = (\underbrace{0, \dots, 0}_{j-1}, \overset{j}{1}, \underbrace{0, \dots, 0}_{d-j+1}) \in \mathbb{N}^d$. Integrating by parts we find

$$\widehat{(D^\alpha \varphi)}(k) = \int_{\mathbb{R}^d} (\partial_{x_j} \varphi(x)) e^{-i 2\pi k \cdot x} dx$$

$$= -\int_{\mathbb{R}^d} \varphi(x) \partial_{x_j} e^{-i 2\pi k \cdot x} dx = 2\pi i k_j \hat{\varphi}(k).$$

The general case follows by induction in each component.
To prove 2) one shows, in a similar way,

$$(2\pi i)^{|\alpha| - |\beta|} k^\alpha D^\beta \hat{\varphi}(k) = \left(\widehat{D^\alpha((\cdot)^\beta \varphi)}\right)(k).$$

Then

$$\|\hat{\varphi}\|_{\alpha, \beta} = \sup_k |k^\alpha D^\beta \hat{\varphi}(k)| = \|(\cdot)^\alpha D^\beta \hat{\varphi}\|_{L^\infty(\mathbb{R}^d)}$$

$$\lesssim \|\widehat{(\cdot)^\alpha D^\beta \hat{\varphi}}\|_{L^1(\mathbb{R}^d)} = \|D^\alpha((\cdot)^\beta \varphi)\|_{L^1(\mathbb{R}^d)}.$$

Using the product rule several times, the right hand side is a finite sum of integrals of the type

$$\int_{\mathbb{R}^d} |x|^\gamma |D^\delta \varphi(x)| dx \leq \int \frac{1}{1+|x|^M} (1+|x|^M) |x|^\gamma |D^\delta \varphi(x)| dx$$

$$< +\infty. \quad \blacksquare$$

Definition 2.12 (Tempered distribution)

A tempered distribution on $\mathbb{R}^d$ is a continuous linear functional on $S(\mathbb{R}^d)$, i.e., a linear map $T : S(\mathbb{R}^d) \to \mathbb{C}$ such that, if $\varphi_n \in S(\mathbb{R}^d)$ converges to $\varphi \in S(\mathbb{R}^d)$ in $S(\mathbb{R}^d)$, then $\lim_{n\to\infty} T(\varphi_n) = T(\varphi)$.

We denote by $S'(\mathbb{R}^d)$ the set of tempered distributions. Furthermore, we say that a sequence $T_n \in S'(\mathbb{R}^d)$ converges to T in $S'(\mathbb{R}^d)$ if

$$\lim_{n\to\infty} T_n(\varphi) = T(\varphi) \qquad \forall \varphi \in S(\mathbb{R}^d).$$

Having slightly enlarged the space of test functions and, correspondingly the associated notion of convergence of test functions, the space $S'(\mathbb{R}^d)$ is smaller than $\mathcal{D}'(\mathbb{R}^d)$.

Notable examples of tempered distributions:

- T_f for $f \in L^p(\mathbb{R}^d)$ $\quad 1 \le p \le +\infty$.
- T_f for f equal to any complex polynomial.

What belongs to $\mathcal{D}'(\mathbb{R}^d)$ without belonging to $S'(\mathbb{R}^d)$? The problem is in the decay at ∞: we are authorized to test against functions that decay faster than any polynomial but not faster than this. In particular, the function $f(x) = e^{x^2}$ certainly belongs to $L^1_{loc}(\mathbb{R})$, and therefore defines a corresponding $T_f \in \mathcal{D}'(\mathbb{R})$. However, for

$$\varphi(x) = e^{-\frac{1}{2}x^2} \qquad (\text{notice that } \varphi \in S(\mathbb{R}^d)),$$

the integral $f\varphi$ is not summable. Thus $T_f \notin S'(\mathbb{R})$.

Tempered distributions are sometimes called slowly growing.

Definition 2.13 (Fourier transform in $S'(\mathbb{R}^d)$)

Let $T \in S'(\mathbb{R}^d)$. We define its Fourier transform $\hat{T} \in S'(\mathbb{R}^d)$ by its action $\hat{T}(\varphi) := T(\hat{\varphi})$ on any $\varphi \in S(\mathbb{R}^d)$.

Let us check that this indeed defines a tempered distribution. Linearity is obvious. To prove continuity, let $\varphi_n \to \varphi$ in $S(\mathbb{R}^d)$. We need to show that $\hat{\varphi}_n \to \hat{\varphi}$ again in $S(\mathbb{R}^d)$. For any $\alpha, \beta \in \mathbb{N}^d$, we have

$$\|\hat{\varphi}_n - \hat{\varphi}\|_{\alpha,\beta} = \sup_x |k^\beta D^\alpha(\hat{\varphi}_n - \hat{\varphi})| \leq \|D^\beta((\cdot)^\alpha(\varphi_n - \varphi))\|_{L^1(\mathbb{R}^d)}.$$

For the right hand side one again uses the product rule to obtain a finite linear combination of $\|\varphi_n - \varphi\|_{\gamma,\delta}$, which converge to zero by assumption.
Thus,

$$\hat{T}(\varphi_n) = T(\hat{\varphi}_n) \xrightarrow[n\to\infty]{} T(\hat{\varphi}) = \hat{T}(\varphi),$$

which proves continuity.

It is worth mentioning that the above definition agrees with the usual one when $T = T_f$ for some $f \in L^p(\mathbb{R}^d)$ with $1 \leq p \leq 2$. But this gives a meaning to $\hat{T}_f$ if, e.g., $f \in L^\infty(\mathbb{R}^d)$. For the special case of

$$f : \mathbb{R}^d \to \mathbb{C}, \quad f(x) = e^{2\pi i p \cdot x} \quad \text{(plane wave, or complex exponential)}$$

we have $\hat{T}_f = \delta_p$, Dirac delta at p.

As a last topic in this section, and a very important application of the theory of distributions, we introduce the concept of Green function.

Assume we want to find solutions to the inhomogeneous Laplace equation (Poisson's equation)

$$-\Delta u = f \qquad u, f : \mathbb{R}^n \to \mathbb{C}$$

for some fixed, regular enough, f. One might be tempted to write, formally, something like

$$\text{"} u = (-\Delta)^{-1} f \text{"},$$

which might be tricky to properly define. One proper way of writing a sort of inverse of the Laplace operator is the Green function formalism.

We look for a solution to the distributional equation

$$-\Delta G_y = \delta_y \qquad \text{in } \mathcal{D}'(\mathbb{R}^d)$$

for some $y \in \mathbb{R}^d$. What is the meaning of this? The right hand side is the Dirac delta distribution centered at y. The left hand side is the weak laplacian of the unknow distribution $G_y \in \mathcal{D}'(\mathbb{R}^d)$. We mark the y depencence since the r.h.s. depends on y. The above equation actually means

$$(-\Delta G_y)(\phi) = \delta_y(\phi) \equiv \phi(y) \qquad \text{for all } \phi \in \mathcal{D}(\mathbb{R}^d).$$

$$\begin{array}{l} \parallel \\ G_y(-\Delta\phi) \underset{\substack{\uparrow \\ \text{if } G_y \text{ is a function}}}{=} \int_{\mathbb{R}^d} G_y(x)\left(-\Delta\phi(x)\right) dx . \end{array}$$

It turns out that the above equation can be solved, and G_y is a function of the type $G_y(x) = G(x-y)$. Let's discuss for a moment the relevance of this. Define

$$u(x) := \int_{\mathbb{R}^d} G_y(x) f(y)\, dy = G_x(f)$$

This is well-defined at least if $f \in \mathcal{D}(\mathbb{R}^d)$. Then, at least formally,

$$-\Delta u(x) = -\Delta_x \int_{\mathbb{R}^d} G(y) f(x+y)\, dy = -\int_{\mathbb{R}^d} G(y) (\Delta f)(x+y)\, dy$$

$$= -\int G(x-y) (\Delta f)(y)\, dy = -(\Delta G_x)(f) = (\delta_x)(f) = f(x).$$

This will be made more precise in exercise classes. For the moment we simply notice that knowledge of $G_y(x)$ seems to provide a winning way of writing the solution, and that $G_y(x)$ very much behaves like a "inverse" of the laplacian, in the sense that

$$U(x) = \int G_y(x) f(y)\,dy \;\text{"}=\; \int (-\Delta^{-1})(x,y) f(y)\,dy.\text{"}$$

We now show that the functions

$$G_y(x) = -\frac{1}{S^1} \ln|x-y| \qquad d=2$$

$$G_y(x) = \frac{1}{(d-2)\, S^{d-1}} \frac{1}{|x-y|^{d-2}} \qquad d \neq 2$$

are the Green's function for Poisson's equation in $\mathbb{R}^d$. Here

$$|S^d| = 2\pi^{d/2}\,\Gamma\left(\frac{d}{2}\right)$$

is the volume of the unit sphere $S^{d-1} \subset \mathbb{R}^d$.

Theorem 2.14 (Laplacian of Green's functions)

We have $\quad -\Delta G_y = \delta_y \quad$ in $\mathcal{D}'(\mathbb{R}^d)$.

Proof

Without loss of generality we restrict to $y=0$. We have to show, for every $\phi \in \mathcal{D}(\mathbb{R}^d)$,

$$I := \int_{\mathbb{R}^d} (\Delta\phi)(x)\, G_0(x)\,dx = -\phi(0).$$

But from its definition, $G_0 \in L^1_{loc}(\mathbb{R}^d)$, and therefore we only have to show

$$\lim_{r\to 0} I(r) = -\phi(0)$$

with

$$I(r) := \int_{|x|>r} (\Delta\phi)(x)\, G_0(x)\, dx .$$

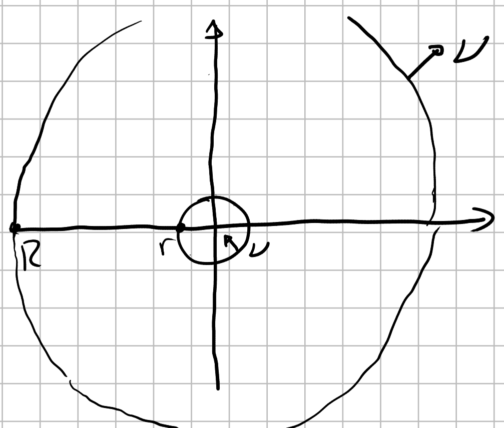
Indeed, the remainder $\int_{|x|<r} (\Delta\phi)(x)\, G_0(x) dx$ vanishes as $r \to 0$ by dominated convergence, since G_0 is L^1_{loc} and $\Delta\phi$ is smooth and bounded.

We can also restrict ourselves, for free, to

$$I(r) = \int_{r<|x|<R} (\Delta\phi)(x)\, G_0(x)\, dx \quad \text{for some } R>0$$

since ϕ has compact support. Now, for $|x|>0$ $G_0(x)$ is in $C^\infty(\mathbb{R}^d)$ and $\Delta G_0 = 0$ (exercise!). We can thus integrate by parts

$$I(r) = \int_{r<|x|<R} (\Delta\phi)(x)\, G_0(x)\, dx = \int_{|x|=r} G_0\, \nabla\phi \cdot \nu\, d\Sigma + \int_{|x|=R} G_0 \underbrace{\nabla\phi}_{0} \cdot \nu\, d\Sigma$$



$$- \int_{r<|x|<R} \nabla\phi(x) \cdot \nabla G_0(x)\, dx$$

$$= \int_{|x|=r} G_0 \nabla\phi \cdot \nu\, d\Sigma - \int_{|x|=r} \phi\, \nabla G_0 \cdot \nu\, d\Sigma$$

$$- \int_{r<|x|<R} \phi(x)\, \underbrace{\Delta G_0(x)}_{0}\, dx$$

$$= \int_{|x|=r} G_0 \nabla\phi \cdot \nu\, d\Sigma - \int_{|x|=r} \phi\, \nabla G_0 \cdot \nu\, d\Sigma .$$

Now, by direct calculation, on $|x|=r$ we have

$$\nabla G_0 \cdot \nu = \frac{1}{|S^{d-1}|} r^{-d+1},$$

and therefore

$$- \int_{|x|=r} \phi\, \nabla G_0 \cdot \nu\, d\Sigma = - \frac{1}{|S^{d-1}|} \int_{|x|=r} \phi(\omega)\, r^{-d+1}\, d\Sigma_\omega = - \frac{1}{|S^{d-1}|} \int_{|x|=1} \phi(r\omega)\, d\Sigma_\omega$$

For $r \to 0$ this converges to $-\phi(0)$ by dominated convergence, since ϕ is continuous. The other integral vanishes since

- $|\nabla\phi \cdot \nu| \leq C$ on $|x| = r$ since ϕ is in $C_c^\infty(\mathbb{R}^d)$.
- $G_0(x) = \frac{1}{(d-2)S^{d-1}} \frac{1}{r^{d-2}}$ for $|x| = r$, and therefore
- $\left| \int_{|x|=r} G_0 \nabla\phi \cdot \nu \, d\Sigma \right| \leq C_d \int_{|x|=r} \frac{1}{r^{d-2}} d\Sigma \leq C_d' \frac{r^{d-1}}{r^{d-2}} = C_d' r \to 0$ as $r \to 0$.

Notice that the second • is different for $d=2$, but we leave this as exercise. We have thus proven

$$\lim_{r \to 0} I(r) = -\phi(0),$$

which concludes the proof. ∎