Chapter 2 : Mean-field theory

2.1. The Curie-Weiss model

In the previous Chapter we obtained some understanding of the phase transition in the Ising model. Our key results followed from understanding sufficiently high and sufficiently low temperatures. But what about temperatures close to the critical point (where the phase transition happens)? In this Ising model this is a very difficult question, whose answer lies well beyond the scope of this lecture.

As an alternative we study the simpler Curie-Weiss model, for which we will be able to answer such kind of questions. The model is defined as the Ising model on the complete graph, that is, every spin interacts with all other spins with the same strength. Because of this, there is no spatial structure in the model. The Hamiltonian on N spins reads

$$\mathcal{H} = \underbrace{- \frac{\beta}{2N} \sum_{x,y=1}^{N} \sigma_x \sigma_y}_{= \frac{\beta}{2} \sum_x \sigma_x \underbrace{\frac{1}{N} \sum_y \sigma_y}_{m \text{ "=" mean field}}} - h \sum_{x=1}^{N} \sigma_x \quad , \quad \sigma \in \{\pm 1\}^N$$

$$= - \frac{\beta N}{2} m^2 - hNm \tag{1}$$

and the Gibbs distribution is given by

$$P^N_{\beta,h}(\omega) = \frac{1}{Z^N_{\beta,h}} \exp\big(-\mathcal{H}(\omega)\big). \tag{2}$$

The CW model is one of the simplest exactly solvable statistical mechanics models.

We start our analysis of the CW model with the following lemma, which is an example of a **Hubbard-Stratanovich transformation.**(*)

Lemma 1: For $\beta \geq 0$ and $h \in \mathbb{R}$, we have

$$Z_{\beta,h}^{N} = \sqrt{\frac{\beta N}{2\pi}} \int_{\mathbb{R}} e^{-NS(\varphi)}\, d\varphi \quad \text{with}$$

$$S(\varphi) = \frac{\beta}{2}\varphi^2 - \ln\left(\cosh\left(\beta\varphi + h\right)\right). \tag{3}$$

Proof: For any $t \in \mathbb{R}$ we have

$$\sqrt{2\pi} = \int_{\mathbb{R}} \exp\left(-\tfrac{1}{2}\varphi^2\right) d\varphi = \int_{\mathbb{R}} \exp\left(-\tfrac{1}{2}\left(\varphi - t\right)^2\right) d\varphi . \tag{4}$$

Hence,

$$\exp\left(\tfrac{1}{2}t^2\right) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left(-\tfrac{1}{2}\varphi^2 + \varphi t\right) d\varphi . \quad (*) \tag{5}$$

For the partition function of the CW model this implies

$$Z^N_{\beta h} = \sum_{\omega \in \{\pm 1\}^N} \underbrace{\exp\left(-\mathcal{H}(\omega)\right)}$$

$$= \underbrace{\exp\left(\frac{\beta N}{2} m^2(\omega)\right)} \exp\left(hN\, m(\omega)\right)$$

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left(-\tfrac{1}{2}\varphi^2 + \sqrt{\beta N}\, m(\omega)\, \varphi\right) d\varphi$$

$$\overset{\varphi \to \varphi\sqrt{\beta N}}{=} \sqrt{\frac{\beta N}{2\pi}} \int_{\mathbb{R}} \exp\left(-\frac{\beta N}{2}\varphi^2 + \beta N\, m(\omega)\, \varphi\right) d\varphi$$

$$= \sqrt{\frac{\beta N}{2\pi}} \int_{\mathbb{R}} \exp\left(-\frac{\beta N}{2}\varphi^2\right) \sum_{\omega \in \{\pm 1\}^N} \exp\Big(N(\beta\varphi + h)\, m(\omega)\Big)\, d\varphi . \tag{6}$$

Next, we write

$$\sum_{\omega\in\{\pm 1\}^N} \exp\Big(N(\beta\varphi+h)\underbrace{m(\omega)}_{\frac{1}{N}\sum_{x=1}^{N}\sigma_x(\omega)} \Big)$$

$$= \sum_{\omega\in\{\pm 1\}^N} \prod_{x=1}^{N} \exp\Big((\beta\varphi+h)\,\sigma_x(\omega) \Big)$$

$$= \prod_{x=1}^{N} \underbrace{\sum_{\omega_x=\pm 1} \exp\Big((\beta\varphi+h)\,\omega_x \Big)}_{=\,2\cosh\left((\beta+h)\varphi\right)} = \Big[2\cosh\big(\beta\varphi+h\big)\Big]^N. \qquad (7)$$

In combination, (6) and (7) imply

$$Z^N_{\beta,h} = \sqrt{\frac{\beta N}{2\pi}} \int_{\mathbb{R}} \exp\Big(-\frac{\beta N}{2}\varphi^2\Big)\Big[2\cosh\big(\beta\varphi+h\big)\Big]^N d\varphi, \qquad (8)$$

which proves (3). $\blacksquare$

From the lemma we learn that the sum over spins can be replaced by an integral, which is easier to analyse.

To analyse the integral in the large N limit, we apply Laplace's method. The precise result we need is captured in the following lemma.

Lemma 2: Assume that $S: \mathbb{R} \to \mathbb{R}$ is a continuous function, which attains its (not necessarily unique) minimum at $\varphi_0 \in \mathbb{R}$, and satisfies $\int_{\mathbb{R}} \exp(-S(\varphi))\, d\varphi < +\infty$ as well as $|\{\varphi : S(\varphi) \leq \min S + 1\}| < +\infty$. Then we have

$$\lim_{N \to \infty} \ln\left(\int_{\mathbb{R}} \exp(-NS(\varphi))\, d\varphi \right) = -S(\varphi_0). \qquad (9)$$

Proof: Step 1 : Lower bound

We have

$$\int_{\mathbb{R}} \exp\big(-NS(\varphi)\big)\, d\varphi \geq \int_{\{|S(\varphi)-\min S|<\delta\}} \exp\Big(-N\big(S(\varphi_0)+\varepsilon\big)\Big)\, d\varphi$$

Continuity

$$\geq \exp\Big(-N\big(S(\varphi_0)+\varepsilon\big)\Big) \cdot \Big|\{|S(\varphi)-\min S|<\delta\}\Big| \tag{10}$$

$< +\infty$ if $0 \leq \delta < 1$

This implies

$$\liminf_{N\to\infty} \frac{1}{N} \ln\left(\int_{\mathbb{R}} \exp\big(-NS(\varphi)\big)\, d\varphi\right)$$

$$\geq -S(\varphi_0) - \varepsilon \tag{11}$$

for any $\varepsilon > 0$, and hence also with $\varepsilon = 0$.

Step 2: Upper bound

Here we write

$$\int_{\mathbb{R}} \exp\big(-N(S(\varphi))\big)\, d\varphi = \exp\big(-NS(\varphi_0)\big) \underbrace{\int_{\mathbb{R}} \exp\Big(-N\big(\underbrace{S(\varphi)-S(\varphi_0)}_{\geq 0}\big)\Big)\, d\varphi}_{\leq \int_{\mathbb{R}} \exp\left(-\left(S(\varphi)-S(\varphi_0)\right)\right) d\varphi}$$

$$\leq \exp(-NS(\varphi_0)) \exp\left(S(\varphi_0)\right) \underbrace{\int_{\mathbb{R}} \exp(-S(\varphi))\, d\varphi}_{<+\infty \text{ by assumption}} . \tag{12}$$

We apply ln on both sides, divide by N, take $\limsup_{N\to\infty}$ and find

$$\limsup_{N\to\infty} \frac{1}{N} \ln\left(\int_{\mathbb{R}} \exp(-N(S(\varphi))\, d\varphi \right) \leq -S(\varphi_0). \tag{13}$$

In combination with (11) this proves the claim of the lemma.

$\blacksquare$

The specific free energy of the CW model is defined by

$$f_N(\beta,h) = -\frac{1}{N} \ln\left(Z^N_{\beta,h}\right), \tag{14}$$

the magnetization reads

$$m_N(\beta,h) = \langle \sigma_1 \rangle_{\beta,h}^N = \sum_{\omega \in \{\pm 1\}^N} \sigma_1(\omega)\, P_{\beta,h}^N(\omega) \tag{15}$$

$$= -\frac{\partial f_N(\beta,h)}{\partial h}$$

and the magnetic susceptibility is given by

$$\chi_N(\beta,h) = \sum_{x=1}^{N} \underbrace{\langle \sigma_1 ; \sigma_x \rangle_{\beta,h}^N}_{\langle \sigma_1 \sigma_x \rangle_{\beta,h}^N - \left(\langle \sigma_1 \rangle_{\beta,h}^N\right)^2} = \frac{\partial m_N(\beta,h)}{\partial h} . \tag{16}$$

The following statement is a direct consequence of Lemma 1 and 2.

Theorem 1: For all $\beta \geq 0$ and $h \in \mathbb{R}$, the following limits exist:

(a) $$\lim_{N \to \infty} f_N(\beta,h) = \min \mathcal{S} =: f(\beta,h), \tag{17}$$

(b) $$m(\beta,h) = \lim_{N\to\infty} m_N(\beta,h) \quad \text{at } (\beta,h), \text{ where } f \text{ is differentiable} \tag{18}$$

(c) $$m_+(\beta) = \lim_{h\to 0} m(\beta,h) \quad \text{(spontaneous magnetization).} \tag{19}$$

Moreover, the spontaneous magnetization satisfies

$$m_+(\beta) = \begin{cases} >0 & \text{if } \beta > \beta_c \\ =0 & \text{if } \beta \leq \beta_c \end{cases} \quad \text{with } \beta_c = 1 \tag{20}$$

as well as

$$\lim_{\beta \downarrow \beta_c} \frac{m_+(\beta)}{3(\beta-\beta_c)^{1/2}} = 1. \tag{21}$$

Before we give the proof of the above theorem, we state the following lemma, which will be proved in the exercises.

Lemma 3: The following statements are true:

(a) For any $\beta \geq 0$ and $h \neq 0$, S has a unique global minimum φ_0 of the same sign as h.

(b) For $\beta \underset{(<)}{\leq} 1$ and $h \in \mathbb{R}$, S is (strictly) convex and thus has a unique global minimum, which tends to 0 when $h \to 0$.

(c) For $\beta > 1$ and $h = 0$, S has two global minima $\pm\varphi_0 \neq 0$ and as $h \to \pm 0$ the unique global min tends to $\pm\varphi_0 \neq 0$.

(d) The global minimum is differentiable in h when $h \neq 0$ or $\beta < 1$.

Remark 1. The minima of $S(\varphi) = \frac{\beta}{2}\varphi^2 - \ln\left(\cosh\left(\beta\varphi + h\right)\right)$ satisfy the equation

$$S'(\varphi) = \beta\varphi - \beta \tanh\left(\beta\varphi + h\right) = 0$$

$$\iff \quad \varphi = \tanh\left(\beta\varphi + h\right), \tag{22}$$

which is sometimes called the **self-consistent equation**.

Proof of Thm. 1: As a direct consequence of Lemma 1 and 2 (please check the assumptions!) we have

$$\lim_{N\to\infty} f_N(\beta,h) = \min \mathcal{S}. \tag{23}$$

Next, we note that the susceptibility can be written as

$$\chi_N(\beta,h) = \mathrm{Var}_{\beta,h}(m_N) = \langle m_N^2\rangle_{\beta,h} - \left(\langle m_N\rangle_{\beta,h}\right)^2 \geq 0. \tag{24}$$

That is, $\frac{\partial^2 f_N}{\partial h^2} \leq 0$ and f_N is concave. We already proved in the exercises that if $f_n(x) \to f(x)$ for all $x \in \mathbb{R}$ and f is convex, then $f_n'(x) \to f'(x)$ at every point x, where f is differentiable. Accordingly,

$$m_N(\beta,h) \xrightarrow{N\to\infty} m(\beta,h) \tag{25}$$

holds at such points.

An application of Lemma 3 shows

$$m_+(\beta) = \lim_{h\downarrow 0} m(\beta,h) = \lim_{h\downarrow 0}\left[-\frac{\partial f(\beta,h)}{\partial h}\right]$$

$$= -\lim_{h\downarrow 0}\frac{\partial}{\partial h}\underbrace{\left[\frac{\beta}{2}\varphi_0^2(\beta,h) - \ln\Big(\cosh\big(\varphi_0(\beta,h)\beta+h\big)\Big)\right]}_{S(\varphi_0,\beta,h)}$$

φ_0: unique solution to self-consistent equation at $h>0$

$$\left[\;\frac{dS}{dh}(\varphi_0,\beta,h) = \Big(\underbrace{\frac{\partial S}{\partial\varphi_0}}_{=0}\frac{\partial\varphi_0}{\partial h}\Big)(\varphi_0,\beta,h) + \frac{\partial S}{\partial h}(\varphi_0,\beta,h)\right.$$

$$\left. = -\frac{\partial}{\partial h}\ln\Big(\cosh\big(\varphi_0\beta+h\big)\Big) = -\tanh(\varphi_0\beta)\;\right]$$

$$= \lim_{h\downarrow 0}\tanh\big(\varphi_0(\beta,h)+h\big)$$

$$= \begin{cases} 0 & \text{if } \beta\le 1 \\ >0 & \text{if } \beta>1. \end{cases} \tag{26}$$

To see how $m_+(\beta)$ tends to 0 as $\beta \downarrow 1$, we note that

$$\tanh(\varphi_0 \beta) = \varphi_0 \beta - \frac{1}{3}(\varphi_0 \beta)^3 + O((\varphi_0\beta)^5) \tag{27}$$

for $\varphi_0\beta \to 0$. For the self-consistent equation this implies

$$\varphi_0(1-\beta) = \frac{1}{3}(\varphi_0\beta)^3 + o((\varphi_0\beta)^3). \tag{28}$$

Since $\varphi_0(\beta, 0+) > 0$ for $\beta > 1$, we can divide by φ_0 and find

$$3(1-\beta) = \varphi_0^2 \beta^3 (1 + o(1))$$

$$\Rightarrow \varphi_0(\beta, 0+) \simeq \sqrt{3(1-\beta)} \quad \text{for } \beta \downarrow 1 \tag{29}$$

Hence,

$$\begin{aligned} m_+(\beta) &= \tanh(\varphi_0(\beta, 0+)\beta) \\ &\simeq \varphi_0(\beta, 0+) \simeq \sqrt{3(1-\beta)} \quad \text{for } \beta \downarrow 1. \end{aligned} \tag{30}$$

$\blacksquare$

In the exercises you will show: The susceptibility is finite for all $h \in \mathbb{R}$ if $\beta < 1$ and for $h \neq 0$ if $\beta > 1$. Moreover,

$$\chi(\beta, 0) \simeq \frac{1}{\beta_c - \beta} \quad \text{for } \beta \uparrow \beta_c \quad \text{and}$$

$$\chi(\beta, 0+) \simeq \frac{1}{2(\beta - \beta_c)} \quad \text{for } (\beta \downarrow \beta_c). \tag{31}$$

Remark 2: The powers in the behavior of m, m_+ and χ are called critical exponents. Based on scaling invariance at the critical point $(\beta, h) = (\beta_c, 0)$ one very generally expects that

$$\begin{aligned} m_+(\beta) &\simeq A_1 (\beta - \beta_c)^{a} && (\beta \downarrow \beta_c) \\ m(\beta_c, h) &\simeq A_2 h^{1/\delta} && (h \downarrow 0) \\ \chi(\beta, 0) &\simeq A_3 (\beta_c - \beta)^{-\gamma} && (\beta \uparrow \beta_c) \\ &\vdots \end{aligned} \tag{32}$$

For the CW model we have $a = \frac{1}{2}$, $\delta = 3$, $\gamma = 1$.

The Ising model on $\mathbb{Z}^d$ has the same critical exp. as the CW model if $d \geq 5$. The crit. exp. of the Ising model on $\mathbb{Z}^2$ are given by $\alpha = \frac{1}{8}$, $\delta = 15$, $\gamma = 7/4$.

2.2. Mean-field bounds for the Ising model

In this section we write the Hamiltonian of the CW model as

$$\mathcal{H}^{CW}_{N;\beta,h}(\omega) = -\frac{d\beta}{N}\sum_{i,j=1}^{N}\omega_i\omega_j - h\sum_{i=1}^{N}\omega_i, \tag{33}$$

that is, we replaced β by $2d\beta$. Before we had $\beta_c = 1$ and now we have $\beta_c^{CW} = \frac{1}{2d}$. By $\psi_\beta^{CW}(h) = -f(\beta,h)$ we denote the pressure and $m_\beta^{CW}(h)$ is the magnetization.

Then we have

Theorem 2: The following bounds hold for the Ising model on $\mathbb{Z}^d$, $d \geq 1$:

$$\begin{aligned}
&(a)\quad \psi(\beta,h) \geq \psi_\beta^{CW}(h), \text{ for all } \beta \geq 0 \text{ and all } h \in \mathbb{R}\\
&(b)\quad \langle \sigma_0 \rangle^+_{\beta,h} \leq m_\beta^{CW}(h), \text{ for all } \beta \geq 0 \text{ and } h \geq 0\\
&(c)\quad \beta_c(d) \geq \beta_c^{CW}.
\end{aligned} \tag{34}$$